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Uncertainty relation associated with a monotone pair skew information

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ABSTRACT

We derive a trace inequality leading to an uncertainty relation based on the monotone pair skew information introduced by Furuichi. As the monotone pair skew information generalizes the Wigner–Yanase–Dyson skew information as well as some other skew information, our result also extends a few known results on the uncertainty relations. Particularly it reduces to that of Luo, Yanagi, and Furuichi et al. in the special cases.

1. Introduction

The skew information plays important roles in quantum mechanics and quantum information theory. The information is in particular a central tool to understand the uncertainty relation. For a density matrix ρ and an observable A, Wigner–Yanase skew information

$$I_\rho(A) \equiv \frac{1}{2}\mathrm{Tr}\big(\big(i[\rho^{1/2}, A]\big)^2\big) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^{1/2} A \rho^{1/2} A) \tag{1.1}$$

was defined in [11]. Here the commutator is defined by $[A, B] = AB - BA$. This quantity was generalized by Dyson as

$$I_{\rho,\alpha}(A) \equiv \frac{1}{2}\mathrm{Tr}\big(\big(i[\rho^\alpha, A]\big)\big(i[\rho^{1-\alpha}, A]\big)\big) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A), \quad \alpha \in [0,1], \tag{1.2}$$

and it is known as the Wigner–Yanase–Dyson skew information. The relation between these quantities and the uncertainty relation was studied in [3,6,7,9,10,12,13].

Recently in [2], for a monotone pair (f, g) of operator monotone functions, Furuichi introduced the (f, g)-skew information by

$$\begin{aligned} I_{\rho,(f,g)}(A) &\equiv \frac{1}{2}\mathrm{Tr}\big(\big(i[f(\rho), A_0]\big)\big(i[g(\rho), A_0]\big)\big) \\ &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2\big) - \mathrm{Tr}\big(f(\rho)A_0 g(\rho)A_0\big) \end{aligned} \tag{1.3}$$

where for an observable A, $A_0 \equiv A - \mathrm{Tr}(\rho A)I$, I an identity operator. For $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha}$ $(0 < \alpha < 1)$, $I_{\rho,(f,g)}$ reduces to $I_{\rho,\alpha}$ in (1.2). For this information, Furuichi has shown the following trace inequality [2]:

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$$I_{\rho,(f,g)}(A) I_{\rho,(f,g)}(B) \geqslant \left|\mathrm{Re}\left\{\mathrm{Corr}_{\rho,(f,g)}(A,B)\right\}\right|^2, \tag{1.4}$$

where the (f,g)-correlation measure $\mathrm{Corr}_{\rho,(f,g)}(A,B)$ is defined by

$$\mathrm{Corr}_{\rho,(f,g)}(A,B) = \mathrm{Tr}\big(f(\rho)g(\rho)A^*B\big) - \mathrm{Tr}\big(f(\rho)A^*g(\rho)B\big).$$

The purpose of this paper is to obtain an uncertainty relation based on the above monotone pair skew information. We show that for all monotone pairs (f,g) of operator monotone functions, which are compatible in logarithmic increase (CLI monotone pair, in short), the Yanagi-type uncertainty relation holds (see Section 2 for the details). Since there are many CLI monotone pairs this result extends some of the existing results. For instance, the uncertainty relations of Yanagi [12], Luo [9], and Furuichi et al. [3] follow as special cases. We remark that recently Gibilisco and Isola attempted another generalization by using quantum Fisher information [5].

We would like to mention that it is strongly desirable to investigate the convexity of the monotone pair skew information in order that it is really a physically meaningful information measure. The convexity of the Wigner–Yanase–Dyson skew information was shown by Lieb [8]. We don't discuss, however, this matter in this paper.

This paper is organized as follows. In Section 2, we briefly review some uncertainty relations and state the main result (Theorem 2.1). Then we give some examples which show that it extends the existing results. Section 3 is devoted to the proof. The key is to find a lower bound away from zero of some functional of CLI monotone pair (Proposition 3.1).

2. Trace inequalities and main result

We start by introducing some uncertainty relations. Let M_n (resp. $M_{n,sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). Let D_n be the set of strictly positive elements of M_n while $D_n^1 \subset D_n$ is the set of strictly positive density matrices, that is,

$$D_n^1 = \left\{\rho \in M_n \mid \mathrm{Tr}(\rho) = 1,\ \rho > 0\right\}.$$

Let $\rho \in D_n^1$ be fixed. For any $A \in M_{n,sa}$, define $A_0 \equiv A - \mathrm{Tr}(\rho A)I$, where $I \in M_n$ is the identity matrix. For the density matrix ρ, the expectation of A is expressed by $\mathrm{Tr}(\rho A)$ and the variance for ρ and A is defined by $V_\rho(A) \equiv \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2 = \mathrm{Tr}(\rho A_0^2)$.

The most famous uncertainty relation, which is also known as uncertainty principle in quantum mechanics, is that of Heisenberg's: for any $\rho \in D_n^1$ and $A, B \in M_{n,sa}$,

$$V_\rho(A)V_\rho(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^2. \tag{2.1}$$

Further strong result was given by Schrödinger:

$$V_\rho(A)V_\rho(B) - \frac{1}{4}\left|\mathrm{Tr}\big(\rho\{A_0,B_0\}\big)\right|^2 \geqslant \frac{1}{4}\left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^2. \tag{2.2}$$

Here the anticommutator on M_n is defined by $\{A_0, B_0\} = A_0B_0 + B_0A_0$. Among further generalizations, we introduce here the results of Luo [9], Yanagi [12], and Furuichi et al. [3]. For the results of Luo and Yanagi, which are one parameter extended versions of the inequality (2.1), we introduce the Wigner–Yanase–Dyson skew information $I_{\rho,\alpha}(A)$ and some other quantities:

$$I_{\rho,\alpha}(A) \equiv \mathrm{Tr}\big(\rho A_0^2\big) - \mathrm{Tr}\big(\rho^\alpha A_0 \rho^{1-\alpha} A_0\big), \tag{2.3}$$

$$J_{\rho,\alpha}(A) \equiv \mathrm{Tr}\big(\rho A_0^2\big) + \mathrm{Tr}\big(\rho^\alpha A_0 \rho^{1-\alpha} A_0\big), \tag{2.4}$$

$$U_{\rho,\alpha}(A) \equiv \sqrt{I_{\rho,\alpha}(A) J_{\rho,\alpha}(A)}. \tag{2.5}$$

Yanagi gave the following uncertainty relation on Wigner–Yanase–Dyson skew information [12]:

$$U_{\rho,\alpha}(A) U_{\rho,\alpha}(B) \geqslant \alpha(1-\alpha)\left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^2. \tag{2.6}$$

It is a generalization to Luo's, which is the case for $\alpha = 1/2$ [9,10]. It is worth noticing that the inequality (2.6) does not hold in general if we replace $U_{\rho,\alpha}$ by $I_{\rho,\alpha}$ [3,10].

Furuichi et al. considered another generalization of Wigner–Yanase skew information and the associated uncertainty relation in the following way. Define for $0 \leqslant \alpha \leqslant 1$,

$$K_{\rho,\alpha}(A) \equiv \frac{1}{2}\mathrm{Tr}\left[\left(i\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]\right)^2\right],$$

$$L_{\rho,\alpha}(A) \equiv \frac{1}{2}\mathrm{Tr}\left[\left(\left\{\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right\}\right)^2\right], \tag{2.7}$$

and

$$W_{\rho,\alpha}(A) \equiv \sqrt{K_{\rho,\alpha}(A)L_{\rho,\alpha}(A)}.$$

Then the uncertainty relation

$$W_{\rho,\alpha}(A)W_{\rho,\alpha}(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^{2}[A,B]\right]\right|^{2} \tag{2.8}$$

holds. See Theorem 2.6 of [3].

From now on we focus on the monotone pair skew information introduced by Furuichi [2]. Recall that a function f on the real line is called an operator monotone function if for any operators A and B with $A \leqslant B$, the inequality $f(A) \leqslant f(B)$ holds. Typical examples are x^{α} and $(\frac{x}{1+x})^{\alpha}$ for $\alpha \in (0,1)$. See [4] and references therein. Recall the Furuichi's (f,g)-skew information $I_{\rho,(f,g)}$ for a monotone pair (f,g) in (1.3). When we take $f(x)=x^{\alpha}$ and $g(x)=x^{1-\alpha}$, $I_{\rho,(f,g)}$ reduces to $I_{\rho,\alpha}$ in (2.3). Also, if we take $f(x)=g(x)=\frac{x^{\alpha}+x^{1-\alpha}}{2}$, $I_{\rho,(f,g)}$ reduces to $K_{\rho,\alpha}$ in (2.7). Thus, monotone pair skew information gives a rich class of generalizations to Wigner–Yanase skew information, including Wigner–Yanase–Dyson skew information. As mentioned in Introduction, Furuichi obtained a trace inequality (1.4). What we would like to do is to find an uncertainty relation associated to monotone pair skew information. For that purpose we need to impose further conditions on the pair.

Definition 2.1. Let $f(x)$ and $g(x)$ be nonnegative operator monotone functions defined on the interval $[0,1]$. We call the pair (f,g) a compatible in log-increase, monotone pair (CLI monotone pair, in short) if

(a) $(f(x)-f(y))(g(x)-g(y)) \geqslant 0$ for all $x,y \in [0,1]$,
(b) $f(x)$ and $g(x)$ are differentiable on $(0,1)$ and

$$0 < \inf_{0<x<1}\frac{G'(x)}{F'(x)} \leqslant \sup_{0<x<1}\frac{G'(x)}{F'(x)} < \infty,$$

where $F(x)=\log f(x)$ and $G(x)=\log g(x)$.

Example 2.1. Let $f(x)=x^{\alpha}$, $g(x)=x^{1-\alpha}$, $0<\alpha<1$. Then

$$\frac{G'(x)}{F'(x)} = \frac{1-\alpha}{\alpha}, \quad \forall x \in (0,1).$$

So, (f,g) is a CLI monotone pair.

For each CLI monotone pair (f,g), we introduce the correlation functions in the following way.

Definition 2.2. Let $\rho \in D_n^1$ and (f,g) be a CLI monotone pair. For $A,B \in M_{n,sa}$, we define

$$\begin{aligned} I_{\rho,(f,g)}(A,B) &\equiv \frac{1}{2}\mathrm{Tr}\big(\big(i\big[f(\rho),A_0\big]\big)\big(i\big[g(\rho),B_0\big]\big)\big) \\ &= \frac{1}{2}\mathrm{Tr}\big(f(\rho)g(\rho)\{A_0,B_0\} - \big(f(\rho)A_0g(\rho)B_0 + f(\rho)B_0g(\rho)A_0\big)\big), \end{aligned} \tag{2.9}$$

$$\begin{aligned} J_{\rho,(f,g)}(A,B) &\equiv \frac{1}{2}\mathrm{Tr}\big(\big\{f(\rho),A_0\big\}\big\{g(\rho),B_0\big\}\big) \\ &= \frac{1}{2}\mathrm{Tr}\big(f(\rho)g(\rho)\{A_0,B_0\} + \big(f(\rho)A_0g(\rho)B_0 + f(\rho)B_0g(\rho)A_0\big)\big) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} I_{\rho,(f,g)}(A) &\equiv I_{\rho,(f,g)}(A,A) \\ &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2 - f(\rho)A_0g(\rho)A_0\big), \end{aligned} \tag{2.11}$$

$$\begin{aligned} J_{\rho,(f,g)}(A) &\equiv J_{\rho,(f,g)}(A,A) \\ &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2 + f(\rho)A_0g(\rho)A_0\big), \end{aligned} \tag{2.12}$$

$$U_{\rho,(f,g)}(A) \equiv \sqrt{I_{\rho,(f,g)}(A)J_{\rho,(f,g)}(A)}. \tag{2.13}$$

Note that $I_{\rho,(f,g)}(A) \geqslant 0$. See (3.3) and the proof of Theorem 2 of [1]. When we take $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha}$, $0 < \alpha < 1$, it holds that $J_{\rho,(f,g)} = J_{\rho,\alpha}$ and $U_{\rho,(f,g)} = U_{\rho,\alpha}$, likewise $I_{\rho,(f,g)} = I_{\rho,\alpha}$. We are now ready to state our main result. For each CLI monotone pair (f, g) we let

$$\beta_{(f,g)} \equiv \min\left\{\frac{m}{(1+m)^2}, \frac{M}{(1+M)^2}\right\}, \tag{2.14}$$

where $m = \inf_{0<x<1} \frac{G'(x)}{F'(x)}$ and $M = \sup_{0<x<1} \frac{G'(x)}{F'(x)}$. Notice that $\beta_{(g,f)} = \beta_{(f,g)}$.

Theorem 2.1. *Let (f, g) be a CLI monotone pair. For any $\rho \in D_n^1$, the following inequality holds*:

$$U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) \geqslant \beta_{(f,g)}\left|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\right|^2, \quad A, B \in M_{n,sa}. \tag{2.15}$$

The proof of Theorem 2.1 is given in Section 3.

Remark 2.1. (a) Let $f(x) = x^\alpha$ and $g(x) = x^\gamma$, $\alpha, \gamma \in (0,1)$ on $[0,1]$. The pair (f, g) is a CLI monotone pair and $\beta_{(f,g)} = \frac{\alpha\gamma}{(\alpha+\gamma)^2}$. In particular, if $\gamma = 1-\alpha$, then $\beta_{(f,g)} = \alpha(1-\alpha)$ and the relation (2.15) reduces to (2.6) (Yanagi's result of [12]).

(b) Let $f(x) = g(x) = \frac{x^\alpha + x^{1-\alpha}}{2}$, $\alpha \in (0,1)$ on $[0,1]$. In this case we have $\beta_{(f,g)} = \frac{1}{4}$ and the relation (2.15) reduces to (2.8), the result of Furuichi et al. [3].

(c) We may have much more examples. For instance, any pair of the functions $(\frac{x}{1+x})^\alpha$ with different α's, and even the pair (f, g) with $f(x) = (\frac{x}{1+x})^\alpha$ and $g(x) = x(1+x)\log(1+\frac{1}{x})$ (g is an operator monotone function, see [4]) are CLI monotone pairs.

3. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. We adopt a similar method used in Section 3 of [12].

Let $\rho = \sum_l \lambda_l |\phi_l\rangle\langle\phi_l| \in D_n^1$, where $\{\phi_l\}_{l=1}^n$ is an orthonormal set in $\mathbb{C}^n$, that is, $\mathrm{Tr}(\rho) = \sum_l \lambda_l = 1$, $\lambda_l > 0$. Let (f, g) be a CLI monotone pair. By a simple calculation, we have

$$\begin{aligned}
\mathrm{Tr}\big(f(\rho)g(\rho)A_0^2\big) &= \sum_l f(\lambda_l)g(\lambda_l)\langle A_0\phi_l, A_0\phi_l\rangle \\
&= \sum_{l,m} f(\lambda_l)g(\lambda_l)\langle\phi_m, A_0\phi_l\rangle\langle A_0\phi_l, \phi_m\rangle \\
&= \sum_{l,m} f(\lambda_l)g(\lambda_l)|a_{ml}|^2 \\
&= \sum_{l,m} \frac{f(\lambda_l)g(\lambda_l) + f(\lambda_m)g(\lambda_m)}{2}|a_{ml}|^2
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\mathrm{Tr}\big(f(\rho)A_0g(\rho)A_0\big) &= \sum_{l,m} f(\lambda_l)g(\lambda_m)\langle\phi_m, A_0\phi_l\rangle\langle A_0\phi_l, \phi_m\rangle \\
&= \sum_{l,m} f(\lambda_l)g(\lambda_m)|a_{ml}|^2 \\
&= \sum_{l,m} \frac{f(\lambda_l)g(\lambda_m) + f(\lambda_m)g(\lambda_l)}{2}|a_{ml}|^2
\end{aligned} \tag{3.2}$$

for any $A \in M_{n,sa}$, where $a_{ml} = \langle\phi_m, A_0\phi_l\rangle$ and $a_{lm} = \overline{a_{ml}}$.

From (3.1) and (3.2) we get

$$\begin{aligned}
I_{\rho,(f,g)}(A) &= \sum_{l,m} \frac{(f(\lambda_l)g(\lambda_l) + f(\lambda_m)g(\lambda_m)) - (f(\lambda_l)g(\lambda_m) + f(\lambda_m)g(\lambda_l))}{2}|a_{ml}|^2 \\
&= \sum_{l<m} \Big[\big(f(\lambda_l)g(\lambda_l) + f(\lambda_m)g(\lambda_m)\big) - \big(f(\lambda_l)g(\lambda_m) + f(\lambda_l)g(\lambda_m)\big)\Big]|a_{ml}|^2 \\
&= \sum_{l<m} \big(f(\lambda_l) - f(\lambda_m)\big)\big(g(\lambda_l) - g(\lambda_m)\big)|a_{ml}|^2,
\end{aligned} \tag{3.3}$$

$$\begin{aligned} J_{\rho,(f,g)}(A) &= \sum_{l,m} \frac{(f(\lambda_l)g(\lambda_l)+f(\lambda_m)g(\lambda_m))+(f(\lambda_l)g(\lambda_m)+f(\lambda_m)g(\lambda_l))}{2}|a_{ml}|^2 \\ &\geqslant \sum_{l<m}\big(\big(f(\lambda_l)g(\lambda_l)\big)+f(\lambda_m)g(\lambda_m)\big)+\big(f(\lambda_l)g(\lambda_m)+f(\lambda_m)g(\lambda_l)\big)|a_{ml}|^2 \\ &= \sum_{l<m}\big(f(\lambda_l)+f(\lambda_m)\big)\big(g(\lambda_l)+g(\lambda_m)\big)|a_{ml}|^2. \end{aligned} \tag{3.4}$$

To prove Theorem 2.1, we need to control a lower bound of a functional coming from a CLI monotone pair. For a given CLI monotone pair (f,g), define a function L on $[0,1]\times[0,1]$ by

$$\begin{aligned} L(x,y) &= \frac{(f(x)g(x)+f(y)g(y))^2-(f(x)g(y)+f(y)g(x))^2}{(f(x)g(x)-f(y)g(y))^2} \\ &= \frac{(f(x)^2-f(y)^2)(g(x)^2-g(y)^2)}{(f(x)g(x)-f(y)g(y))^2}. \end{aligned} \tag{3.5}$$

It is easy to show that $L(x,y)\leqslant 1$ for all $x,y\in[0,1]$.

Proposition 3.1. *For any CLI monotone pair (f,g), we have*

$$\min_{x,y\in[0,1]} L(x,y)\geqslant 4\beta_{(f,g)}$$

where $\beta_{(f,g)}$ is defined in (2.14).

For the proof of Proposition 3.1 we need the following lemma.

Lemma 3.1. *For $0<a,b$ and any real number r, the inequality*

$$\frac{(e^{2ar}-1)(e^{2br}-1)}{(e^{(a+b)r}-1)^2}\geqslant\frac{4ab}{(a+b)^2}$$

holds.

Proof. Without loss of generality, we assume $r\geqslant 0$. Expanding into a power series, we have

$$\begin{aligned} \frac{(e^{2ar}-1)(e^{2br}-1)}{(e^{(a+b)r}-1)^2} &= \frac{4abr^2+\sum_{n=3}2^n((a+b)^n-a^n-b^n)\frac{r^n}{n!}}{(a+b)^2r^2+\sum_{n=3}(2^n-2)(a+b)^n\frac{r^n}{n!}} \\ &= \frac{4ab}{(a+b)^2}\cdot\frac{1+\sum_{n=3}A_n\frac{r^{n-2}}{n!}}{1+\sum_{n=3}B_n\frac{r^{n-2}}{n!}} \\ &\geqslant \frac{4ab}{(a+b)^2} \end{aligned}$$

where $A_n=\frac{2^n}{4ab}((a+b)^n-a^n-b^n)$ and $B_n=(2^n-2)(a+b)^{n-2}$. We can show $A_n\geqslant B_n$ by a straightforward induction argument, which utilizes the convexity of x^n when $n\geqslant 1$. $\square$

Proof of Proposition 3.1. We let $m=\inf_{0<x<1}\frac{G'(x)}{F'(x)}$ and $M=\sup_{0<x<1}\frac{G'(x)}{F'(x)}$, so $\beta_{(f,g)}=\min\{\frac{m}{(1+m)^2},\frac{M}{(1+M)^2}\}$. Without loss of generality, we assume that both $f(x)$ and $g(x)$ are increasing. Let $x<y$. In the last line of (3.5), dividing both the numerator and the denominator by $(f(x)g(x))^2$ and by using $F(x)=\log f(x)$ and $G(x)=\log g(x)$, we get

$$L(x,y)=\frac{(e^{2(F(y)-F(x))}-1)(e^{2(G(y)-G(x))}-1)}{(e^{F(y)-F(x)+G(y)-G(x)}-1)^2}. \tag{3.6}$$

By the generalized mean value theorem, there exists a z, $x<z<y$, such that $\frac{G(y)-G(x)}{F(y)-F(x)}=\frac{G'(z)}{F'(z)}=:r(z)$. Thus we have

$$L(x,y)=\frac{(e^{2(F(y)-F(x))}-1)(e^{2r(z)(F(y)-F(x))}-1)}{(e^{(1+r(z))(F(y)-F(x))}-1)^2}. \tag{3.7}$$

Notice that for any $R>1$, the function $r\mapsto A(r):=\frac{(R^2-1)(R^{2r}-1)}{(R^{1+r}-1)^2}$ defined in the interval $m\leqslant r\leqslant M$ is bounded from below by

$\min\{A(m), A(M)\}$.

Thus since $m \leqslant r(z) \leqslant M$ we get

$$L(x, y) \geqslant \min\left\{\frac{(e^{2F(x,y)}-1)(e^{2mF(x,y)}-1)}{(e^{(1+m)F(x,y)}-1)^2}, \frac{(e^{2F(x,y)}-1)(e^{2MF(x,y)}-1)}{(e^{(1+M)F(x,y))}-1)^2}\right\}, \tag{3.8}$$

where $F(x, y) \equiv F(y) - F(x) > 0$. Applying Lemma 3.1 to (3.8) we complete the proof. □

Proof of Theorem 2.1. Since

$$\begin{aligned}\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big) &= \mathrm{Tr}\big(f(\rho)g(\rho)[A_0,B_0]\big)\\ &= 2i\,\mathrm{Im}\big[\mathrm{Tr}\big(f(\rho)g(\rho)A_0B_0\big)\big]\\ &= 2i\,\mathrm{Im}\sum_{l<m}\big(f(\lambda_l)g(\lambda_l)-f(\lambda_m)g(\lambda_m)\big)\langle\phi_m, A_0\phi_l\rangle\langle B_0\phi_l,\phi_m\rangle\\ &= 2i\sum_{l<m}\big(f(\lambda_l)g(\lambda_l)-f(\lambda_m)g(\lambda_m)\big)\,\mathrm{Im}(a_{ml}b_{lm})\end{aligned}$$

for any $A \in M_{n,sa}$, where $a_{ml} = \langle\phi_m, A_0\phi_l\rangle$ and $b_{ml} = \langle\phi_m, B_0\phi_l\rangle$, we have

$$\begin{aligned}\big|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\big| &\leqslant 2\sum_{l<m}\big|f(\lambda_l)g(\lambda_l)-f(\lambda_m)g(\lambda_m)\big|\big|\mathrm{Im}[a_{ml}b_{lm}]\big|\\ &\leqslant 2\sum_{l<m}\big|f(\lambda_l)g(\lambda_l)-f(\lambda_m)g(\lambda_m)\big||a_{ml}||b_{lm}|.\end{aligned} \tag{3.9}$$

By Proposition 3.1, we have

$$\begin{aligned}\beta_{(f,g)}\big|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\big|^2 &\leqslant 4\beta_{(f,g)}\left(\sum_{l<m}\big|f(\lambda_l)g(\lambda_l)-f(\lambda_m)g(\lambda_m)\big||a_{ml}b_{lm}|\right)^2\\ &\leqslant \left(\sum_{l<m}\sqrt{\big(f(\lambda_l)^2-f(\lambda_m)^2\big)\big(g(\lambda_l)^2-g(\lambda_m)^2\big)}|a_{ml}b_{lm}|\right)^2.\end{aligned}$$

Substituting $(f(\lambda_l)^2 - f(\lambda_m)^2)(g(\lambda_l)^2 - g(\lambda_m)^2) = (f(\lambda_l)g(\lambda_l) + f(\lambda_m)g(\lambda_m))^2 - (f(\lambda_l)g(\lambda_m) + f(\lambda_m)g(\lambda_l))^2$ into the above, and by the Schwarz inequality, we have

$$\begin{aligned}&\beta_{(f,g)}\big|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\big|^2\\ &\quad\leqslant \sum_{l<m}\big[\big(f(\lambda_l)g(\lambda_l)+f(\lambda_m)g(\lambda_m)\big)-\big(f(\lambda_l)g(\lambda_m)+f(\lambda_m)g(\lambda_l)\big)\big]|a_{ml}|^2\\ &\qquad\times\sum_{l<m}\big[\big(f(\lambda_l)g(\lambda_l)+f(\lambda_m)g(\lambda_m)\big)+\big(f(\lambda_l)g(\lambda_m)+f(\lambda_m)g(\lambda_l)\big)\big]|b_{lm}|^2.\end{aligned} \tag{3.10}$$

Then by (3.3) and (3.4) we have

$$\beta_{(f,g)}\big|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\big|^2 \leqslant I_{\rho,(f,g)}(A)J_{\rho,(f,g)}(B),$$

and similarly

$$\beta_{(f,g)}\big|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\big|^2 \leqslant I_{\rho,(f,g)}(B)J_{\rho,(f,g)}(A).$$

Hence by multiplying the above two inequalities, we have

$$\beta_{(f,g)}\big|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)\big|^2 \leqslant U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B). \qquad \square$$

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