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A Variational Principle in the Dual Pair of Reproducing Kernel Hilbert Spaces and an Application

Hyun Jae Yoo[1]

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Given a positive definite, bounded linear operator A on the Hilbert space $\mathcal{H}_0 := l^2(E)$, we consider a reproducing kernel Hilbert space $\mathcal{H}_+$ with a reproducing kernel $A(x, y)$. Here E is any countable set and $A(x, y)$, $x, y \in E$, is the representation of A w.r.t. the usual basis of $\mathcal{H}_0$. Imposing further conditions on the operator A, we also consider another reproducing kernel Hilbert space $\mathcal{H}_-$ with a kernel function $B(x, y)$, which is the representation of the inverse of A in a sense, so that $\mathcal{H}_- \supset \mathcal{H}_0 \supset \mathcal{H}_+$ becomes a rigged Hilbert space. We investigate the ratios of determinants of some partial matrices of A and B. We also get a variational principle on the limit ratios of these values. We apply this relation to show the Gibbsianness of the determinantal point process (or fermion point process) defined by the operator $A(I + A)^{-1}$ on the set E.

KEY WORDS: Reproducing kernel Hilbert space, determinantal point process, Gibbs measure, interaction

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1. INTRODUCTION

In this paper we will consider certain variational principle arising in the dual pair of reproducing kernel Hilbert spaces (abbreviated RKHS's hereafter). Then we will find an application in showing the Gibbsianness of some determinantal point processes (in short DPP's) in discrete spaces.

Let E be any countable set, e.g., $E = \mathbb{Z}^d$, the d-dimensional lattice space. Let $\mathcal{H}_0 := l^2(E)$ be the Hilbert space of square summable functions (sequences) on E. Let A be a bounded and positive definite operator on $\mathcal{H}_0$, and let $A(x, y), x, y \in E$,

[1] Department of Applied Mathematics, Hankyong National University, 67 Sukjong-dong, Ansung-city, Gyonggi-do 456-749, Korea; e-mail: yoohj@hknu.ac.kr.

be the matrix elements of A w.r.t. the usual basis. We emphasize that A may have 0 in its spectrum. Imposing suitable conditions on A, we will get another matrix elements $B(x, y)$, which is roughly the representation of A^{-1}, though A^{-1} may be an unbounded operator. We let $\mathcal{H}_+$ and $\mathcal{H}_-$ be the RKHS's with reproducing kernels (abbreviated RK's) $A(x, y)$ and $B(x, y)$, respectively.

The variational principle we will address is about the ratios of determinants of the partial matrices of $A(x, y)$ and $B(x, y)$. Equivalently, it is about the projections of a fixed vector in the subspaces of $\mathcal{H}_-$ and $\mathcal{H}_+$ (Theorem 2.4).

The variational principle will be applied to show the Gibbsianness of a DPP defined by the operator $A(I + A)^{-1}$. In fact, the variational principle will guarantee the existence of global Papangelou intensity (Theorem 2.6). This proves the Gibbsianness of the DPP and we will give a proper interaction potential, and show also the uniqueness of the Gibbs measure (Theorem 2.7).

We remark here that the main idea in showing the Gibbsianness has been borrowed from Ref. 13. We should, however, point out that since the operators A dealt with in Ref. 13 are strictly positive, there is a severe restriction in applications. For example, if A is a diagonal matrix with diagonal elements $\alpha_x > 0$ that decrease to zero as $x \to \infty$ (we let $E = \mathbb{Z}$ or $\mathbb{N}$), then the DPP corresponding to the operator $A(I + A)^{-1}$ is clearly a Gibbs measure for the system with potential energy

$$V(\xi) = -\sum_{x\in\xi} \log \alpha_x, \quad \xi \text{ finite.} \tag{1.1}$$

Even this kind of simple example lies outside the regime of Ref. 13. This paper improves Ref. 13 (in regard of Gibbsianness of DPP's) in that our setting includes more general classes as well as the above example.

At the time of submission of this paper the author was informed that Shirai and Takahashi also obtained a similar result for the variational principle, Theorem 2.4.[14] They introduced quadratic forms for the dual relation.

This paper is organized as follows. In Sec. 2, we introduce the basics of the RKHS's (Sec. 2.1) and DPP's (Sec. 2.2), and then give the main results (Sec. 2.3). Section 3 is devoted to the proof of variational principle, Theorem 2.4. In Sec. 4, we first prove the existence of the global Papangelou intensity, Theorem 2.6. Then we prove the Gibbsianness and its uniqueness, Theorem 2.7. In the Appendix, we provide with some examples.

2. PRELIMINARIES AND MAIN RESULTS

In this Section we review some basics of RKHS's and DPP's. Then we state the main results of the paper.

2.1. Reproducing Kernel Hilbert Spaces

For our convenience, we start from a Hermitian positive definite bounded linear operator A on the complex Hilbert space $\mathcal{H}_0 := l^2(E)$ equipped with the usual inner product

$$(f, g)_0 := \sum_{x \in E} \overline{f(x)} g(x), \quad f, g \in \mathcal{H}_0. \tag{2.1}$$

Here E is any countable set. Throughout this paper we assume that the kernel space of A is trivial: $\ker A = \{0\}$. Then, since $\overline{\operatorname{ran} A} = (\ker A^*)^\perp = (\ker A)^\perp = \mathcal{H}_0$, the range of A is dense in $\mathcal{H}_0$. Let $\mathsf{B} = \{e_x : x \in E\}$ be the usual basis of $\mathcal{H}_0$, i.e., $e_x \in \mathcal{H}_0$ is the unit vector whose component is one at x and zero at all other sites. Let $A(x, y)$, $x, y \in E$, be the matrix elements of the operator A w.r.t. the basis B:

$$A(x, y) := (e_x, Ae_y)_0, \quad x, y \in E. \tag{2.2}$$

On the dense subspace $\operatorname{ran} A$, we define a new inner product as

$$(f, g)_+ := (f, A^{-1} g)_0, \quad f, g \in \operatorname{ran} A. \tag{2.3}$$

Denote by $\| \cdot \|_+$ the resulting norm and let $\mathcal{H}_+$ be the completion of $\operatorname{ran} A$ w.r.t. $\| \cdot \|_+$. We notice that $\mathcal{H}_+$ is a RKHS[1,10] with kernel function $A(x, y)$, that is, the following defining conditions are satisfied:

(i) For every $x \in E$, the function $A(\cdot, x)$ belongs to $\mathcal{H}_+$,
(ii) The reproducing property: for every $x \in E$ and $g \in \mathcal{H}_+$, $g(x) = (A(\cdot, x), g)_+$.

Let us now consider another Hilbert space $\mathcal{H}_-$ which is the closure of $\mathcal{H}_0$ w.r.t. the norm $\| \cdot \|_-$ induced by the inner product:

$$(f, g)_- := (f, Ag)_0, \quad f, g \in \mathcal{H}_0. \tag{2.4}$$

By the boundedness of A we have the inclusions:

$$\mathcal{H}_- \supset \mathcal{H}_0 \supset \mathcal{H}_+. \tag{2.5}$$

We want to see $\mathcal{H}_-$ also as a RKHS. It is important to notice that though $\mathcal{H}_0$ may be understood as a class of functions defined on the set E, the completed space $\mathcal{H}_-$ may not be a space of functions defined on the same space E. This is so called a *functional completion* problem,[1] which we now discuss.

Let F be a space of functions on E which is a pre-Hilbert space. By a functional completion of F, as introduced in Ref. [1, p. 347], we mean a completion of F by adjunction of functions on E such that the evaluation map at any site $y \in E$ is a continuous function on the completed space. The following theorem proved by Aronszajn[1] gives a necessary and sufficient condition for the functional completion.

Theorem 2.1. (Aronszajn) *Let* F *be a class of functions on* E *forming a pre-Hilbert space. In order that there exists a functional completion of* F*, it is necessary and sufficient that*

(i) for every fixed $y \in E$*, the linear functional* $f(y)$ *defined in* F *is continuous;*
(ii) for a Cauchy sequence $\{f_n\} \subset \mathsf{F}$*, the condition* $f_n(y) \to 0$ *for every* y *implies that* f_n *itself converges to* 0 *in norm.*

If the functional completion is possible, it is unique.

In our setting, the incomplete class of functions is $\mathcal{H}_0$ equipped with the inner product $(\cdot, \cdot)_-$. We shall demand $\mathcal{H}_-$ to be functionally completed, and now we state all the conditions we need as a hypothesis:

(H) The Hermitian positive definite linear operator A on $\mathcal{H}_0$ is bounded and satisfies (i) $\ker A = \{0\}$; (ii) $\mathcal{H}_-$ is functionally completed.

In the Appendix we will consider some examples of the operators A that satisfy the conditions in (H).

It turns out that a dual pairing between $\mathcal{H}_-$ and $\mathcal{H}_+$ plays very useful roles in characterizing the RK's of $\mathcal{H}_-$ and $\mathcal{H}_+$ as well as in many computations in the sequel. For $f \in \mathcal{H}_0$ and $g \in \operatorname{ran} A$, define

$$ {}_-\langle f, g\rangle_+ := \sum_{x \in E} \overline{f(x)} g(x). \tag{2.6} $$

We have then the bound $|{}_-\langle f, g\rangle_+| \le \|f\|_- \|g\|_+$. Since $\mathcal{H}_0$ and $\operatorname{ran} A$ are dense respectively in $\mathcal{H}_-$ and $\mathcal{H}_+$, the dual pairing extends continuously to a bilinear form on $\mathcal{H}_- \times \mathcal{H}_+$, for which we use the same notation ${}_-\langle f, g\rangle_+$, $f \in \mathcal{H}_-$ and $g \in \mathcal{H}_+$, and the bound also continues to hold:

$$ |{}_-\langle f, g\rangle_+| \le \|f\|_- \|g\|_+, \quad f \in \mathcal{H}_-, \ g \in \mathcal{H}_+. \tag{2.7} $$

For a convenience, we also define its conjugate bilinear form

$$ {}_+\langle g, f\rangle_- := \overline{{}_-\langle f, g\rangle_+}, \quad f \in \mathcal{H}_-, \ g \in \mathcal{H}_+. \tag{2.8} $$

Notice that for $f \in \mathcal{H}_0$, $Af \in \mathcal{H}_+$ and

$$ \|Af\|_+^2 = (Af, A^{-1}Af)_0 = \|f\|_-^2. \tag{2.9} $$

Thus, A extends to an isometry between $\mathcal{H}_-$ and $\mathcal{H}_+$. We will denote the extension by the same A and its inverse by A^{-1}. The following results say some of the usefulness of the duality.

Proposition 2.2. *Assume the hypothesis (H). Then the following properties hold.*

(a) For any $g \in \mathcal{H}_+$*, the functional* ${}_-\langle \cdot, g\rangle_+$ *on* $\mathcal{H}_-$ *has norm* $\|g\|_+$*.*
(b) For any $f \in \mathcal{H}_-$*, the functional* ${}_-\langle f, \cdot\rangle_+$ *on* $\mathcal{H}_+$ *has norm* $\|f\|_-$*.*
(c) $\mathcal{H}_+$ *and* $\mathcal{H}_-$ *are the dual spaces of each other.*

(d) *For any $y \in E$, $e_y \in \mathcal{H}_+$. More concretely, $e_y \in \mathcal{H}_+$ is equivalent to saying that the functional $f(y)$ is continuous on $\mathcal{H}_-$.*

Proof. The proofs of (a) and (b) are obvious. (c) By the isometries $A : \mathcal{H}_- \to \mathcal{H}_+$ and its inverse $A^{-1} : \mathcal{H}_+ \to \mathcal{H}_-$, it is easy to check that

$$_-\langle \cdot, g\rangle_+ = (\cdot, A^{-1}g)_- \text{and}\, _-\langle f, \cdot\rangle_+ = (Af, \cdot)_+, \quad f \in \mathcal{H}_-,\ g \in \mathcal{H}_+. \tag{2.10}$$

That is, $\mathcal{H}_+$ and $\mathcal{H}_-$ are respectively the dual spaces of each other via the dual pairing $_-\langle \cdot, \cdot\rangle_+$. (d) If $e_y \in \mathcal{H}_+$, then obviously $f(y) = {}_-\langle f, e_y\rangle_+$ is continuous on $\mathcal{H}_-$. On the other hand, suppose that the functional $f(y)$ is continuous on $\mathcal{H}_-$. Then, by (c), there is a unique element $l_y \in \mathcal{H}_+$ such that

$$f(y) = {}_+\langle l_y, f\rangle_-, \quad f \in \mathcal{H}_-. \tag{2.11}$$

Since finitely supported vectors f are dense in $\mathcal{H}_-$ and for those vectors f we have $_+\langle l_y, f\rangle_- = \sum_x \overline{l_y(x)} f(x)$, l_y must be e_y. □

Finally, we notice that since for any fixed $y \in E$ the functionals $\mathcal{H}_- \ni f \mapsto f(y)$ and $\mathcal{H}_+ \ni g \mapsto g(y)$ are continuous, respectively in $\mathcal{H}_-$ and $\mathcal{H}_+$, it is obvious that

$$_-\langle f, g\rangle_+ = \sum_{x \in E} \overline{f(x)} g(x), \text{ if either } f \text{ or } g \text{ is locally supported.} \tag{2.12}$$

2.2. Determinantal Point Processes on Discrete Sets

Determinantal point processes, or fermion random point fields, are probability measures on the configuration space of, say, particles. The particles may move on the continuum spaces or on the discrete spaces. In this paper we will focus on the DPP's on the discrete sets.

The basics of DPP's including their definitions and basic properties can be found in several papers.[3–6,12,13,15] We will review the definition of DPP's mainly from Ref 13. Let E be a countable set and let K be a Hermitian positive definite bounded linear operator on the Hilbert space $\mathcal{H}_0 = l^2(E)$. Let $\mathcal{X}$ be the configuration space on E, that is, $\mathcal{X}$ is the class of all subsets of E. We frequently understand a point $\xi = (x_i)_{i=1,2,\cdots} \in \mathcal{X}$ as a configuration of particles located at the sites $x_i \in E$, $i = 1, 2, \ldots$. The following is an existence theorem for DPP's. We state it as appeared in Ref. 13.

Theorem 2.3. *Let E be a countable discrete space and K be a Hermitian bounded operator on $\mathcal{H}_0 = l^2(E)$. Assume that $0 \le K \le I$. Then, there exists a unique probability Borel measure μ on $\mathcal{X}$ such that for any finite subset $X \subset E$,*

$$\mu(\{\xi \in \mathcal{X} : \xi \supset X\}) = \det(K(x, y))_{x,y \in X}. \tag{2.13}$$

The σ-algebra on $\mathcal{X}$ is induced from the product topology on $\{0, 1\}^E$ (see Sec. 4). Here we remark that the left hand side of (2.13) is just the correlation function of the probability measure μ, thus the theorem says that the correlation functions of DPP's are given by the determinants of positive definite kernel functions.

The most useful feature in the theory of DPP's is that there can be given an exact formula for the density functions of local marginals. For each subset $\Lambda \subset E$, let P_Λ denote the projection operator on $\mathcal{H}_0$ onto the space of vectors which have supports on the set Λ. Let $K_\Lambda := P_\Lambda K P_\Lambda$ be the restriction of K on the projection space. Given a configuration $\xi \in \mathcal{X}$, we let ξ_Λ be the restriction of ξ on the set Λ, i.e., $\xi_\Lambda := \xi \cap \Lambda$. For each finite subset $\Lambda \subset E$, assuming first that $I_\Lambda - K_\Lambda$ is invertible, we define

$$A_{[\Lambda]} := K_\Lambda (I_\Lambda - K_\Lambda)^{-1}. \tag{2.14}$$

Then for the DPP μ corresponding to the operator K, the local marginals are given by the formula: for each finite subset $\Lambda \subset E$ and fixed $\xi \in \mathcal{X}$,

$$\mu(\{\zeta : \zeta_\Lambda = \xi_\Lambda\}) = \det(I_\Lambda - K_\Lambda) \det(A_{[\Lambda]}(x, y))_{x,y \in \xi_\Lambda}, \tag{2.15}$$

where $A_{[\Lambda]}(x, y)$, $x, y \in \Lambda$, denote the matrix components of $A_{[\Lambda]}$. Though in this paper we will confine ourselves to the case where $A_{[\Lambda]}$ is well-defined as a bounded operator, we remark that the formula (2.15) is meaningful even if K_Λ has 1 in its spectrum.[13,15]

2.3. Results

First we will consider a variational principle for the positive definite operator A introduced in Sec. 2.1. Since we are assuming that $\mathcal{H}_-$ is functionally completed, for any $y \in E$ the functional $f(y)$ is continuous on $\mathcal{H}_-$, or $e_y \in \mathcal{H}_+$ (Proposition 2.2(d)). This condition, on the other hand, is equivalent to the one that $\mathcal{H}_-$ is a RKHS Ref. [1, p. 343]. Let $B(x, y)$ be the RK for $\mathcal{H}_-$. From the reproducing property we see that $B(x, y)$ is the value of the function $A^{-1}e_y$ at x (see Ref. [1, p. 343]), that is

$$B(x, y) = {}_+\langle e_x, A^{-1} e_y \rangle_-, \quad x, y \in E. \tag{2.16}$$

Let $x_0 \in E$ be a fixed point and let $E = \{x_0\} \cup R_1 \cup R_2$ be a partition of E. For each $\Delta \subset E$, we let $\mathsf{F}_{\mathrm{loc},\Delta}$ be the local functions supported on Δ:

$$\mathsf{F}_{\mathrm{loc},\Delta} := \text{the class of finite linear combinations of } \{e_x : x \in \Delta\}. \tag{2.17}$$

In the sequel, we denote by $\Lambda \Subset E$ that Λ is a finite subset of E. We are concerned with the following numbers. For each $\Lambda \Subset E$, define

$$\alpha_\Lambda := \inf_{f \in \mathsf{F}_{\mathrm{loc},\Lambda \cap R_1}} \|e_{x_0} - f\|_-^2 \quad \text{and} \quad \beta_\Lambda := \inf_{g \in \mathsf{F}_{\mathrm{loc},\Lambda \cap R_2}} \|e_{x_0} - g\|_+^2. \tag{2.18}$$

Obviously, both of the sequences of nonnegative numbers $\{\alpha_\Lambda\}_{\Lambda \Subset E}$ and $\{\beta_\Lambda\}_{\Lambda \Subset E}$ decrease as Λ increases. We let

$$\alpha := \lim_{\Lambda \uparrow E} \alpha_\Lambda \quad \text{and similarly} \quad \beta := \lim_{\Lambda \uparrow E} \beta_\Lambda. \tag{2.19}$$

One of the main result of this paper is the following:

Theorem 2.4. *Let the operator A satisfy the conditions in the hypothesis (H). Then the product of the numbers α and β defined in (2.19) is one: $\alpha\beta = 1$.*

We remark that the result of the theorem was obtained by Shirai and Takahashi in the case when the bounded operator A is strictly greater than 0, i.e., $0 < cI \le A$ for some positive constant c.[13]

The following example shows that without condition (Hii), we may have $\alpha = 0$.

Example 2.5. Let $E := \mathbb{N}$, the set of natural numbers. Let $\{e_n\}_{n=1}^\infty$ be the usual basis for $\mathcal{H}_0 := l^2(E)$. Define a positive definite bounded linear operator A on $\mathcal{H}_0$ by $A := B^*B$, where B is defined by

$$Be_n := \begin{cases} e_1, & n = 1, \\ \frac{1}{n}(e_1 + e_n), & n \ge 2, \end{cases}$$

and by a linear extension. Then, it is not hard to show that $\mathcal{H}_-$ is not functionally completed and moreover, for the decomposition $E = \{1\} \cup R_1 \cup R_2$ with $R_1 = E \setminus \{1\}$ and $R_2 = \emptyset$, we have $\alpha = 0$.

Next we apply the above result to show the Gibbsianness of some DPP's. Let A be an operator on $\mathcal{H}_0$ that satisfies the hypothesis (H). Let μ be the DPP corresponding to the operator $K := A(I + A)^{-1}$. Given a fixed point $x_0 \in E$ and a configuration $\xi \in \mathcal{X}$ with $x_0 \notin \xi$, and for each $\Lambda \Subset E$ with $x_0 \in \Lambda$, let $\alpha_{[\Lambda]}$ be the ratio of the probabilities for the configurations $x_0\xi_\Lambda$ and ξ_Λ in Λ:

$$\alpha_{[\Lambda]} := \frac{\mu_\Lambda(x_0\xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda)}, \tag{2.20}$$

where we have simplified the event $\{\zeta \in \mathcal{X} : \zeta_\Lambda = \xi_\Lambda\} \equiv \xi_\Lambda$, etc, and $x_0\xi_\Lambda = \{x_0\} \cup \xi_\Lambda$. By (2.15), $\alpha_{[\Lambda]}$ is computed via the ratio of determinants:

$$\alpha_{[\Lambda]} = \frac{\det A_{[\Lambda]}(x_0\xi_\Lambda, x_0\xi_\Lambda)}{\det A_{[\Lambda]}(\xi_\Lambda, \xi_\Lambda)}, \tag{2.21}$$

where $A_{[\Lambda]}(\xi_\Lambda, \xi_\Lambda) = (A_{[\Lambda]}(x, y))_{x,y \in \xi_\Lambda}$. We are interested in the behavior of the sequence $\{\alpha_{[\Lambda]}\}$ as Λ increases to E. The following theorem gives the answer.

Theorem 2.6. *Let the operator A satisfy the conditions in (H). Then*

$$\lim_{\Lambda \uparrow E} \alpha_{[\Lambda]} = \alpha, \tag{2.22}$$

where α is given in (2.18)–(2.19) *with $R_1 = \xi$ and $R_2 = E \setminus (\xi \cup \{x_0\})$.*

A corollary to this theorem is that the DPP μ corresponding to the operator $A(I + A)^{-1}$ is a Gibbs measure. We state this as a theorem.

Theorem 2.7. *Let the operator A satisfy the conditions in (H). Then the DPP μ corresponding to the operator $A(I + A)^{-1}$ is a Gibbs measure. The interaction potential is given by the logarithm of determinants of submatrices of A: for any finite configuration $\xi \in \mathcal{X}$, the interaction potential $V(\xi)$ is*

$$V(\xi) = -\log \det(A(x, y))_{x,y\in\xi}. \tag{2.23}$$

Moreover, μ is the unique Gibbs measure for the potential energy (2.23).

The above result also extends that obtained in Ref. [13, Theorem 6.2]. We also notice that the idea developed in Refs. 3 and 16, which concerns exclusively with continuum models, can be applied to discrete model and would get some result on the Gibbsianness of μ. The result would look like the following (cf. Ref. [3, Proposition 3.9]): Let $E \equiv \mathbb{Z}^d$ and suppose that (i) A is of finite range in the sense that $A(x, y) = 0$ if $|x - y| \geq R$ for some finite number $R > 0$ and (ii) μ does not percolate. Then μ is a Gibbs measure corresponding to the potential in (2.23). Our result Theorem 2.7 is stronger than this, too.

3. PROOF OF THE VARIATIONAL PRINCIPLE

In this Section we prove Theorem 2.4. The most important tool in the proof is the theory of restrictions and projections in the RKHS's. In Sec. 3.1, we deal with the variational principle in the finite systems. In Sec. 3.2, we first introduce the restriction theory in the RKHS's and then discuss the limit theorems of RK's. Section 3.3 discusses the perturbed norms and their convergence. The proof of Theorem 2.4 is given in Sec. 3.4.

3.1. Variational Principle in the Finite Systems

We discuss the variational principle for positive definite matrices on a finite set. Let $\Lambda \Subset E$ be a finite set and let $(C(x, y))_{x,y\in\Lambda}$ be a positive definite matrix with an inverse C^{-1}. We define two norms on the class F_Λ of functions on Λ as follows:

$$\|f\|_-^2 := \sum_{x,y\in\Lambda} \overline{f(x)}C(x,y)f(y), \quad f \in \mathsf{F}_\Lambda \tag{3.1}$$

and

$$\|g\|_+^2 := \sum_{x,y\in\Lambda} \overline{g(x)}C^{-1}(x,y)g(y), \quad g \in \mathsf{F}_\Lambda. \tag{3.2}$$

Suppose that $\Lambda = \{x_0\} \cup \Lambda_1 \cup \Lambda_2$ is a partition of Λ. Similarly to (2.18) we define

$$a := \inf_{f\in\mathsf{F}_{\Lambda_1}} \|e_{x_0} - f\|_-^2 \quad \text{and} \quad b := \inf_{g\in\mathsf{F}_{\Lambda_2}} \|e_{x_0} - g\|_+^2. \tag{3.3}$$

In the above F_{Λ_i} denotes the class of functions on Λ_i, $i = 1, 2$. By using the elementary calculus and the basic properties of determinants, one can express the numbers a and b concretely with the matrix components. This, as a matter of fact, takes the role of a recipe for the theory in the infinite systems (cf. Ref. [13, Sec. 6]). Below we denote by $C(\Lambda_1, \Lambda_2)$ the submatrix $(C(x,y))_{x\in\Lambda_1,\, y\in\Lambda_2}$ for any subsets Λ_1 and Λ_2 of Λ. We also simplify $\{x\} \cup \Lambda_1$ by $x\Lambda_1$ for $x \notin \Lambda_1$.

Proposition 3.1. *Let $(C(x,y))_{x,y\in\Lambda}$ be a Hermitian positive definite matrix on a finite set Λ with inverse C^{-1}. Let $\Lambda = \{x_0\} \cup \Lambda_1 \cup \Lambda_2$ be a partition of Λ and let the norms $\|\cdot\|_-$ and $\|\cdot\|_+$, and the numbers a and b be defined as in (3.1)–(3.3). Then the following results hold:*

(a) The minimum values a and b are attained respectively at the unique vectors $f_0 = C(\Lambda_1,\Lambda_1)^{-1}C(\Lambda_1, x_0)$ and $g_0 = (C^{-1}(\Lambda_2,\Lambda_2))^{-1}C^{-1}(\Lambda_2, x_0)$:

$$a = \|e_{x_0} - f_0\|_-^2 \quad \text{and} \quad b = \|e_{x_0} - g_0\|_+^2. \tag{3.4}$$

(b)

$$\begin{aligned} a &= \frac{\det C(x_0\Lambda_1, x_0\Lambda_1)}{\det C(\Lambda_1,\Lambda_1)} = (C(x_0\Lambda_1, x_0\Lambda_1)^{-1}(x_0,x_0))^{-1} \\ &= C(x_0,x_0) - C(x_0,\Lambda_1)C(\Lambda_1,\Lambda_1)^{-1}C(\Lambda_1,x_0) \end{aligned} \tag{3.5}$$

and similarly

$$\begin{aligned} b &= \frac{\det C^{-1}(x_0\Lambda_2, x_0\Lambda_2)}{\det C^{-1}(\Lambda_2,\Lambda_2)} = ((C^{-1}(x_0\Lambda_2, x_0\Lambda_2))^{-1}(x_0,x_0))^{-1} \\ &= C^{-1}(x_0,x_0) - C^{-1}(x_0,\Lambda_2)(C^{-1}(\Lambda_2,\Lambda_2))^{-1}C^{-1}(\Lambda_2,x_0). \end{aligned} \tag{3.6}$$

(c) $ab = 1$.

3.2. Restrictions in RKHS's and Limit Theorems of RK's

In this Subsection, we discuss the restriction and projection theories in RKHS's and the limit theorems of RK's. The results we need have been already obtained in Ref. 1. For the readers' convenience, however, we provide it here.

Let us begin with an introduction of the restriction theory in the RKHS's. Suppose that $\mathcal{H}$ is a RKHS with kernel $K(x, y)$, $x, y \in E$. $\mathcal{H}$ might be $\mathcal{H}_-$ or $\mathcal{H}_+$ of our concern. For each subset $\Lambda \subset E$, the function $K_\Lambda(x, y)$, the restriction of $K(x, y)$ to Λ, is still positive definite. The following theorem was proved by Aronszajn (Ref. [1, p. 351]):

Theorem 3.2. *The function $K(x, y)$ restricted to a subset $\Lambda \subset E$ is the reproducing kernel of the class $\mathcal{H}_\Lambda$ of all restrictions of functions in $\mathcal{H}$ to the subset Λ. For any such restriction $f_\Lambda \in \mathcal{H}_\Lambda$, the norm $\|f_\Lambda\|_\Lambda$ is the minimum of $\|f\|$ (the norm of f in $\mathcal{H}$) for all $f \in \mathcal{H}$ whose restrictions to Λ are f_Λ.*

When it is needed to designate the kernel, we use the notations $\mathcal{H}_{\Lambda;K}$ and $\|\cdot\|_{\Lambda;K}$ respectively for the restriction spaces and norms. We remark that the norm $\|f_\Lambda\|_\Lambda$ in Theorem 3.2 is attained at some vector $f' \in \mathcal{H}$ whose restriction is f_Λ:

$$\|f_\Lambda\|_\Lambda = \|f'\| \quad \text{for some } f' \in \mathcal{H} \text{ with } (f')_\Lambda = f_\Lambda. \tag{3.7}$$

We refer to Ref. [1, p. 351] for the details.

Next we discuss the limit theorems of RK's. We will consider two kinds of limits.

A. The case of decreasing sequence. Let $\{E_n\}$ be an increasing sequence of sets with $E = \cup_{n=1}^{\infty} E_n$. For each $n = 1, 2, \ldots$, let F_n be a RKHS defined in E_n with RK $K_n(x, y)$, $x, y \in E_n$. we denote the norm in the space F_n by $\|\cdot\|_n$, $n \geq 1$. For a function $f_n \in \mathsf{F}_n$ we will denote by f_{nm}, $m \leq n$, the restriction of f_n to the set $E_m \subset E_n$. We shall suppose the following two conditions:

(A1) for every $f_n \in \mathsf{F}_n$ and every $m \leq n$, $f_{nm} \in \mathsf{F}_m$;
(A2) for every $f_n \in \mathsf{F}_n$ and every $m \leq n$, $\|f_{nm}\|_m \leq \|f_n\|_n$.

For $m < n$, let F_{nm} be the RKHS consisting of all the restrictions of functions in F_n to E_m (Theorem 3.2). The RK of F_{nm} is $K_{nm}(x, y)$, the restriction of $K_n(x, y)$ to the set E_m, and we denote its norm by $\|\cdot\|_{nm}$. By (A1), $\mathsf{F}_{nm} \subset \mathsf{F}_m$ and by (A2), $\|f_{nm}\|_m \leq \|f_n'\|_n$ for all $f_n' \in \mathsf{F}_n$ with $f_{nm}' = f_{nm}$. Thus by Theorem 3.2, we get $\|f_{nm}\|_m \leq \|f_{nm}\|_{nm}$. Then by Ref. [1, Theorem II of Sec. 7], we have

$$K_{nm} \ll K_m, \quad m < n, \tag{3.8}$$

meaning that $K_m(x, y) - K_{nm}(x, y)$, $x, y \in E_m$, is a positive definite function. The following theorem appears in Ref. [1, Theorem I, Sec. 9]:

Theorem 3.3. *Under the above assumptions on the classes* F_n*, the kernels* K_n *converge to a kernel* $K_0(x, y)$ *defined for all* x, y *in* E*.* K_0 *is the RK of the class* F_0 *of all functions* f_0 *defined in* E *such that*

(i) *their restrictions* f_{0n} *in* E_n *belong to* F_n*,* $n = 1, 2, \ldots$;
(ii) $\lim_{n\to\infty} \|f_{0n}\|_n < \infty$.

The norm of $f_0 \in \mathsf{F}_0$ *is given by* $\|f_0\|_0 = \lim_{n\to\infty} \|f_{0n}\|_n$.

B. The case of increasing sequence. Let $\{E_n\}$ be a decreasing sequence of sets and R be their intersection: $R = \cap_{n=1}^{\infty} E_n$. As in the case A, let F_n, $n = 1, 2\ldots$, be the RKHS's with corresponding kernel functions $K_n(x, y)$, $x, y \in E_n$, $n \geq 1$. As before, we define the restrictions f_{nm} for $f_n \in \mathsf{F}_n$, but now m has to be greater than n. We suppose that F_n form an increasing sequence and the norms $\|\cdot\|_n$ form a decreasing sequence satisfying the following two conditions:

(B1) for every $f_n \in \mathsf{F}_n$ and every $m \geq n$, $f_{nm} \in \mathsf{F}_m$;
(B2) for every $f_n \in \mathsf{F}_n$ and every $m \geq n$, $\|f_{nm}\|_m \leq \|f_n\|_n$.

We then get for the restrictions K_{nm} of K_n the formula

$$K_{nm} \ll K_m, \quad \text{for} \quad m \geq n. \tag{3.9}$$

For each $y \in R$, $\{K_m(y, y)\}$ is an increasing sequence of positive numbers. Its limit may be infinite. We define, consequently,

$$R_0 := \text{the set of } y \in R \text{ such that } K_0(y, y) := \lim_{m\to\infty} K_m(y, y) < \infty. \tag{3.10}$$

Suppose that R_0 is not empty and let F_0 be the class of all restrictions f_{n0} of functions $f_n \in \mathsf{F}_n$ $(n = 1, 2, \ldots)$ to the set R_0. From (B2), the limit $\lim_{k\to\infty} \|f_{nk}\|_k$ exists and we define a norm $\|\cdot\|_0^{\sim}$ on F_0 by[2]

$$\|f\|_0^{\sim} := \inf \lim_{k\to\infty} \|f_{nk}\|_k, \quad f \in \mathsf{F}_0, \tag{3.11}$$

where the infimum is taken over all functions $f_n \in \mathsf{F}_n$, $n \geq 1$, whose restrictions to R_0 are f, i.e., $f(y) = f_{n0}$, $y \in R_0$, for some $f_n \in \mathsf{F}_n$. Now we construct a new space F_0^* and norm $\|\cdot\|_0^*$ on it. Let F_0^* be the class of all functions f_0^* on R_0 such that there is a Cauchy sequence $\{f_0^{(n)}\} \subset \mathsf{F}_0$ satisfying

$$f_0^*(x) = \lim_{n\to\infty} f_0^{(n)}(x), \quad \text{for all } x \in R_0. \tag{3.12}$$

[2] The original definition in Ref. 1 is such that $\|f_{n0}\|_0 := \lim_{k\to\infty} \|f_{nk}\|_k$, but it seems that there is no way to guarantee that $\|f_{n0}\|_0 = \|g_{n0}\|_0$ for different f_n and g_n in F_n with $f_{n0} = g_{n0}$. However, all the arguments in Ref. 1 hold true even if the new norm $\|\cdot\|_0^{\sim}$ in (3.11) is used. In particular, the Theorem 3.4 below holds.

For those vectors f_0^* we define a norm

$$\|f_0^*\|_0^* := \min \lim_{n\to\infty} \|f_0^{(n)}\|_0^{\sim}, \tag{3.13}$$

the minimum being taken over all Cauchy sequences $\{f_0^{(n)}\} \subset \mathsf{F}_0$ satisfying (3.12). There exists at least one Cauchy sequence for which the minimum is attained. Such sequences are called determining f_0^*. The scalar product corresponding to $\|\cdot\|_0^*$ is defined by

$$(f_0^*, g_0^*)_0^* := \lim_{n\to\infty} (f_0^{(n)}, g_0^{(n)})_0^{\sim} \tag{3.14}$$

for any two Cauchy sequences $\{f_0^{(n)}\}$ and $\{g_0^{(n)}\}$ determining f_0^* and g_0^*, respectively. We refer to Ref. [1, Sec. 9] for the details. The following theorem is in Ref. [1, Theorem II, Sec. 9]:

Theorem 3.4. *In the setting of the case B, the restrictions $K_{n0}(x, y)$ for every fixed $y \in R_0$ form a Cauchy sequence in F_0. They converge to a function $K_0^*(x, y) \in \mathsf{F}_0^*$ which is the RK of F_0^*.*

In this paper we will need an application of this theorem for the simplest case described in the following remark.

Remark 3.5. (Ref. [1, p. 368, Remark]) Suppose that the class F_0 with norm $\|\cdot\|_0^{\sim}$ in (3.11) is a subspace of a RKHS F (meaning that $\mathsf{F}_0 \subset \mathsf{F}$ and for all $f \in \mathsf{F}_0$, $\|f\|_0^{\sim}$ is equal to the norm of f in the space F). Then the space F_0^* in Theorem 3.4 is the functional completion of F_0 and $\|\cdot\|_0^*$ is an extension of the norm $\|\cdot\|_0^{\sim}$, and F_0^* is simply the closure of F_0 in F.

3.3. Perturbation of the Norms

In this Subsection, using the results of the previous Subsection, we prove the convergence of norms in the perturbed RKHS's, which will be used in the proof of Theorem 2.4. We first prepare some dual relations in the resticted spaces. Let $R \subset E$ be any subset of E. Following Theorem 3.2, we let $\|\cdot\|_{R;B}$ be the norm of the RKHS $\mathcal{H}_{R;B}$ consisting of all restrictions of vectors in $\mathcal{H}_-$ to the set R and having a RK $B_R(x, y)$, the restriction of $B(x, y)$ to the set R. Here we make a convention. If F is a vector space of functions on any subset $R \subset E$, we sometimes naturally embed it into a vector space of functions on the set E whose restrictions on the set $E \setminus R$ are zero. By abuse of notations, we use the same notation F for the extension, and this should be clear from the context.

Let $(B_R)^{-1}$ be the bounded linear operator on $l^2(R)$ corresponding to the quadratic form[9]

$$Q(f, f) := (f, f)_{R;B}, \quad f \in l^2(R). \tag{3.15}$$

Since $(f, f)_{R;B} \le \|f\|_-^2 = (f, A_R f)_0$, we see that

$$(B_R)^{-1} \le A_R. \tag{3.16}$$

The components of $(B_R)^{-1}$ is given by

$$(B_R)^{-1}(x, y) = (e_x, e_y)_{R;B}, \quad x, y \in R. \tag{3.17}$$

As in the dual relation between $\mathcal{H}_-$ and $\mathcal{H}_+$, we let $\mathcal{H}'_{R;B} \subset l^2(R)$ be the dual space of $\mathcal{H}_{R;B}$: an element $g \in l^2(R)$ belongs to $\mathcal{H}'_{R;B}$ if and only if the (anti-) linear functional

$${}_{R;B}\langle f, g\rangle'_{R;B} := \sum_{x\in R} \overline{f(x)} g(x), \quad f \in l^2(R), \tag{3.18}$$

is continuous w.r.t. $\|\cdot\|_{R;B}$-norm. $\mathcal{H}'_{R;B}$ is none other than the RKHS with RK $(B_R)^{-1}(x, y)$ on R. For each $g \in \mathcal{H}'_{R;B}$ we extend the functional of (3.18) to the whole space $\mathcal{H}_{R;B} \supset l^2(R)$ and keep the dual pairing notation ${}_{R;B}\langle\cdot,\cdot\rangle'_{R;B}$. We denote the norm in $\mathcal{H}'_{R;B}$ by $\|\cdot\|'_{R;B}$. As in the case of the dual pairing ${}_-\langle\cdot,\cdot\rangle_+$, $(B_R)^{-1}$ extends to an isometry from $\mathcal{H}_{R;B}$ onto $\mathcal{H}'_{R;B}$:

$$\|(B_R)^{-1} f\|'_{R;B} = \|f\|_{R;B}, \quad f \in \mathcal{H}_{R;B}. \tag{3.19}$$

It turns out that $\mathcal{H}'_{R;B}$ is a closed subspace of $\mathcal{H}_+$:

Lemma 3.6. *For the spaces $\mathcal{H}_{R;B}$ and $\mathcal{H}'_{R;B}$ the following properties hold.*

(a) $\mathcal{H}'_{R;B} \subset \mathcal{H}_+ \cap l^2(R)$ and for any $g \in \mathcal{H}'_{R;B}$, $\|g\|'_{R;B} = \|g\|_+$.
(b) Let $g \in \mathcal{H}'_{R;B}$. Then for any $f \in \mathcal{H}_-$ that vanishes on R, we have ${}_-\langle f, g\rangle_+ = 0$.

Proof. (a) First we show $\mathcal{H}'_{R;B} \subset \mathcal{H}_+ \cap l^2(R)$. The remaining part will be proved after showing (b). Let $h \in l^2(R)$. Then for any $f \in \mathcal{H}_0 = l^2(E)$, since $(B_R)^{-1}h \in l^2(R)$,

$$\begin{aligned}\left|\sum_{x\in E} \overline{f(x)}(B_R)^{-1}h(x)\right| &= \left|\sum_{x\in R} \overline{f(x)}(B_R)^{-1}h(x)\right| \\ &= |{}_{R;B}\langle f_R, (B_R)^{-1}h\rangle'_{R;B}| \\ &\le \|f_R\|_{R;B}\|(B_R)^{-1}h\|'_{R;B} \\ &\le \|f\|_-\|(B_R)^{-1}h\|'_{R;B},\end{aligned} \tag{3.20}$$

where f_R is the restriction of f to R. Since $\mathcal{H}_0$ is dense in $\mathcal{H}_-$ and $\mathcal{H}_+$ is the dual space of $\mathcal{H}_-$, (3.20) proves that

$$(B_R)^{-1}h \in \mathcal{H}_+ \text{and} \|(B_R)^{-1}h\|_+ \le \|(B_R)^{-1}h\|'_{R;B}. \tag{3.21}$$

Because $l^2(R)$ is dense in $\mathcal{H}_{R;B}$, (3.21) also shows that

$$\mathcal{H}'_{R;B} \subset \mathcal{H}_+ \cap l^2(R). \tag{3.22}$$

(b) Denote by P_R the restriction operator $P_R : \mathcal{H}_- \to \mathcal{H}_{R;B}$ defined by $P_R f := f_R$ for all $f \in \mathcal{H}_-$. Since $\|f_R\|_{R;B} \le \|f\|_-$, the operator P_R is bounded with norm less than or equal to 1. Now let g and f be as in the statement of (b). Let $\{f_n\}$ be any sequence in $\mathcal{H}_0$ that converges to f in $\mathcal{H}_-$. Then since $g \in l^2(R)$, by using the continuity of the operator P_R in $\mathcal{H}_-$, we have

$$\begin{aligned}
{}_-\langle f, g\rangle_+ &= \lim_{n\to\infty} {}_-\langle f_n, g\rangle_+ \\
&= \lim_{n\to\infty} \sum_{x\in R} \overline{f_n(x)} g(x) \\
&= \lim_{n\to\infty} {}_{R;B}\langle P_R f_n, g\rangle'_{R;B} \\
&= {}_{R;B}\langle P_R f, g\rangle'_{R;B} \\
&= 0,
\end{aligned}$$

because $P_R f = 0$.

Finally, we show the last part in (a). Since $(B_R)^{-1} : \mathcal{H}_{R;B} \to \mathcal{H}'_{R;B}$ is an isometry and $l^2(R)$ is dense in $\mathcal{H}_{R;B}$, it is enough to show $\|(B_R)^{-1}h\|'_{R;B} = \|(B_R)^{-1}h\|_+$ for all $h \in l^2(R)$. By (3.21) it remains only to show that $\|(B_R)^{-1}h\|'_{R;B} \le \|(B_R)^{-1}h\|_+$. We have

$$\begin{aligned}
(\|(B_R)^{-1}h\|'_{R;B})^2 &= \|h\|^2_{R;B} \\
&= {}_{R;B}\langle h, (B_R)^{-1}h\rangle'_{R;B} \\
&= {}_-\langle h, (B_R)^{-1}h\rangle_+.
\end{aligned} \tag{3.23}$$

Let $h' \in \mathcal{H}_-$ be the element such that $P_R h' = h$ and $\|h'\|_- = \|h\|_{R;B}$ (see (3.7)). Notice that $h' - h \in \mathcal{H}_-$ vanishes on R and $(B_R)^{-1}h \in \mathcal{H}'_{R;B}$. Thus by (3.23) and the result in (b) we have

$$\begin{aligned}
\|h\|^2_{R;B} &= {}_-\langle h, (B_R)^{-1}h\rangle_+ \\
&= {}_-\langle h', (B_R)^{-1}h\rangle_+ \\
&\le \|h'\|_- \|(B_R)^{-1}h\|_+ \\
&= \|h\|_{R;B} \|(B_R)^{-1}h\|_+.
\end{aligned}$$

This, together with (3.23), proves that $\|(B_R)^{-1}h\|'_{R;B} \le \|(B_R)^{-1}h\|_+$. $\square$

Let us now discuss the perturbation of the operators. Let A be the operator of our concern satisfying the conditions in the hypothesis (H). For each $\varepsilon > 0$ we define new bounded operators as follows:

$$A(\varepsilon) := A + \varepsilon \quad \text{and} \quad B(\varepsilon) := A(\varepsilon)^{-1}, \quad \varepsilon > 0. \tag{3.24}$$

We let $\mathcal{H}_{+;\varepsilon}(:= \mathcal{H}_{E;A(\varepsilon)})$ and $\mathcal{H}_{-;\varepsilon}(:= \mathcal{H}_{E;B(\varepsilon)})$ be the RKHS's with RK's $A(\varepsilon)(x, y)$ and $B(\varepsilon)(x, y)$, respectively. For each $\varepsilon > 0$ we also consider the restriction space $\mathcal{H}_{R;B(\varepsilon)}$ with norm $\|\cdot\|_{R;B(\varepsilon)}$, which is similarly defined as $\mathcal{H}_{R;B}$ by replacing B with $B(\varepsilon)$.

Lemma 3.7. *Let $R \subset E$ be any set. Then for any $f \in l^2(R)$,*

$$\lim_{\varepsilon \to 0} \|f\|_{R;B(\varepsilon)} = \|f\|_{R;B}. \tag{3.25}$$

Proof. First we show that

$$\lim_{\varepsilon \to 0} B(\varepsilon)(x, y) = B(x, y), \quad \text{for all } x, y \in E. \tag{3.26}$$

It is obvious that

$$B(\varepsilon) \ll B(\varepsilon') \quad \text{for } 0 < \varepsilon' < \varepsilon, \tag{3.27}$$

in the sense defined in (3.8). Also, it holds trivially that

$$B(\varepsilon)(y, y) \le B(y, y) < \infty, \quad \forall\, y \in E. \tag{3.28}$$

Moreover, for each fixed $\varepsilon > 0$, since $B(\varepsilon)$ is bounded and strictly positive, the norms $\|\cdot\|_{-;\varepsilon}$ $(:= \|\cdot\|_{E;B(\varepsilon)})$ and $\|\cdot\|_0$ are equivalent on $\mathcal{H}_0 = l^2(E)$. That is, as a set, $\mathcal{H}_{-;\varepsilon}$ is the same as $\mathcal{H}_0$.

It is easy to check that for each $f \in \mathcal{H}_0$, the norm $\|f\|_{-;\varepsilon}$ decreases to $\|f\|_-$ as ε decreases:

$$\lim_{\varepsilon \to 0} \|f\|_{-;\varepsilon} = \|f\|_-. \tag{3.29}$$

Since the norm $\|\cdot\|_-$ is the one for the RKHS with kernel $B(x, y)$, the equality (3.26) follows from Theorem 3.4 (see Remark 3.5).

Let us now prove (3.25). Obviously, for each $f \in l^2(R)$, $\|f\|_{R;B(\varepsilon)}$ decreases as ε decreases and $\|f\|_{R;B(\varepsilon)} \ge \|f\|_{R;B}$ for all $\varepsilon > 0$. Thus the limit

$$\|f\|_0^{\sim} := \lim_{\varepsilon \to 0} \|f\|_{R;B(\varepsilon)}, \quad f \in l^2(R), \tag{3.30}$$

defines a norm on $l^2(R)$. We have to show $\|f\|_0^{\sim} = \|f\|_{R;B}$. Noticing that $\mathcal{H}_{R;B}$ is a RKHS and $\|f\|_0^{\sim} \ge \|f\|_{R;B}$ for $f \in l^2(R)$, we see that for all $y \in R$, the functional $f(y)$ is continuous w.r.t. the $\|\cdot\|_0^{\sim}$-norm on $l^2(R)$. Moreover, the completion $\mathcal{H}_0^{\sim}$

of $l^2(R)$ w.r.t. $\|\cdot\|_0^\sim$ is functionally completed because $\mathcal{H}_0^\sim \subset \mathcal{H}_{R;B}$ and so is $\mathcal{H}_{R;B}$. Therefore, $\mathcal{H}_0^\sim$ is a RKHS [1, p. 343] and $l^2(R)$ is a subspace of it. Since we already know from (3.26) that $B(\varepsilon)_R(x,y) \to B_R(x,y)$ as $\varepsilon \to 0$ for all $x, y \in R$, by Remark 3.5. again, we conclude that $\|\cdot\|_0^\sim$ is the norm for the RKHS on R whose RK is $B_R(x,y)$, i.e., $\|\cdot\|_0^\sim = \|\cdot\|_{R;B}$. The proof is complete. □

3.4. Proof of Theorem 2.4

We are now ready to prove Theorem 2.4. It will be done in a few steps.

Proof of Theorem 2.4.: *Step 1: General facts.* Recall the notation $\mathsf{F}_{\mathrm{loc},\Lambda}$, the class of local functions supported on Λ for the subsets $\Lambda \subset E$. Since $\mathsf{F}_{\mathrm{loc},\Lambda}$ is closed in $\mathcal{H}_-$ for each $\Lambda \Subset E$, there exists a unique element $f_{\Lambda_1,0} \in \mathsf{F}_{\mathrm{loc},\Lambda_1}$ such that

$$\alpha_\Lambda = \inf_{f \in \mathsf{F}_{\mathrm{loc},\Lambda_1}} \|e_{x_0} - f\|_-^2 = \|e_{x_0} - f_{\Lambda_1,0}\|_-^2, \tag{3.31}$$

where $\Lambda_1 := \Lambda \cap R_1$. Let $\mathcal{H}_{1,-}$ be the closure (in $\mathcal{H}_-$) of $\cup_{\Lambda \Subset E} \mathsf{F}_{\mathrm{loc},\Lambda_1} = \mathsf{F}_{\mathrm{loc},R_1}$. Notice that any vector $f \in \mathcal{H}_{1,-}$ vanishes on R_1^c, that is, it is supported on R_1. We also notice that for each $\Lambda \Subset E$, $f_{\Lambda_1,0}$ is the projection of e_{x_0} onto the space $\mathsf{F}_{\mathrm{loc},\Lambda_1}$, and as Λ increases, $f_{\Lambda_1,0}$ converges to the projection of e_{x_0} onto the space $\mathcal{H}_{1,-}$, we call it $f_{R_1,0}$:

$$\lim_{\Lambda \uparrow E} f_{\Lambda_1,0} = f_{R_1,0} \quad (\text{in } \mathcal{H}_-). \tag{3.32}$$

Let us now apply the (extended) operator A to the vector $e_{x_0} - f_{R_1,0}$. We claim that

$$A(e_{x_0} - f_{R_1,0}) = \alpha e_{x_0} + a_2 \in \mathcal{H}_+, \tag{3.33}$$

where the vector $a_2 \in \mathcal{H}_+$ is supported on R_2. In fact, let $A(e_{x_0} - f_{R_1,0}) = a_0 e_{x_0} + a_2 \in \mathcal{H}_+$ with a_2 being supported on $E \setminus \{x_0\}$. Since $f_{R_1,0}$ is the projection of e_{x_0} onto the space $\mathcal{H}_{1,-}$, we have

$$(e_{x_0} - f_{R_1,0}, f)_- = 0 \quad \text{for all } f \in \mathsf{F}_{\mathrm{loc},R_1}. \tag{3.34}$$

Thus, we have for all $f \in \mathsf{F}_{\mathrm{loc},R_1}$,

$$\begin{aligned} 0 &= (e_{x_0} - f_{R_1,0}, f)_- \\ &= {}_+\langle A(e_{x_0} - f_{R_1,0}), f\rangle_- \\ &= {}_+\langle a_0 e_{x_0} + a_2, f\rangle_- \\ &= \sum_{x \in R_1} \overline{a_2(x)} f(x), \end{aligned} \tag{3.35}$$

because f is a local function supported on R_1 (see (2.12)). Since $f \in \mathsf{F}_{\mathrm{loc},R_1}$ is arbitrary, the equation (3.35) proves that a_2 vanishes on R_1. Similarly, it is easily checked that

$$\begin{aligned}\alpha &= \|e_{x_0} - f_{R_1,0}\|_-^2 \\ &= {}_-\langle e_{x_0} - f_{R_1,0}, A(e_{x_0} - f_{R_1,0})\rangle_+ \\ &= \lim_{\Lambda\uparrow E} {}_-\langle e_{x_0} - f_{\Lambda_1,0}, a_0 e_{x_0} + a_2\rangle_+ \\ &= \lim_{\Lambda\uparrow E} a_0 = a_0. \end{aligned} \tag{3.36}$$

We have shown (3.33). Let us now interchange the roles of A, $\|\cdot\|_-$, R_1, and Λ_1 by A^{-1}, $\|\cdot\|_+$, R_2, and Λ_2, respectively. Then we have for each $\Lambda \Subset E$,

$$\beta_\Lambda = \inf_{g \in \mathsf{F}_{\mathrm{loc},\Lambda_2}} \|e_{x_0} - g\|_+^2 = \|e_{x_0} - g_{\Lambda_2,0}\|_+^2, \tag{3.37}$$

for a unique $g_{\Lambda_2,0} \in \mathsf{F}_{\mathrm{loc},\Lambda_2}$, where $\Lambda_2 := R_2 \cap \Lambda$. Also, if we denote by $\mathcal{H}_{2,+}$ the closure of $\cup_{\Lambda \Subset E} \mathsf{F}_{\mathrm{loc},\Lambda_2} = \mathsf{F}_{\mathrm{loc},R_2}$ w.r.t. the $\|\cdot\|_+$-norm, there is a unique $g_{R_2,0} \in \mathcal{H}_{2,+}$ such that

$$\lim_{\Lambda\uparrow E} g_{\Lambda_2,0} = g_{R_2,0} \quad (\text{in } \mathcal{H}_+) \tag{3.38}$$

and

$$\beta = \|e_{x_0} - g_{R_2,0}\|_+^2. \tag{3.39}$$

Similarly to (3.34), we have

$$A^{-1}(e_{x_0} - g_{R_2,0}) = \beta e_{x_0} + b_1 \in \mathcal{H}_-, \tag{3.40}$$

where b_1 is supported on R_1. Now we have on the one hand

$$\begin{aligned}{}_-\langle e_{x_0} - f_{R_1,0}, e_{x_0} - g_{R_2,0}\rangle_+ &= \lim_{\Lambda\uparrow E} {}_-\langle e_{x_0} - f_{\Lambda_1,0}, e_{x_0} - g_{\Lambda_2,0}\rangle_+ \\ &= \lim_{\Lambda\uparrow E} (e_{x_0} - f_{\Lambda_1,0}, e_{x_0} - g_{\Lambda_2,0})_0 \\ &= \lim_{\Lambda\uparrow E} 1 = 1.\end{aligned}$$

On the other hand we have

$$\begin{aligned}1 &= {}_-\langle e_{x_0} - f_{R_1,0}, e_{x_0} - g_{R_2,0}\rangle_+ \\ &= {}_+\langle A(e_{x_0} - f_{R_1,0}), A^{-1}(e_{x_0} - g_{R_2,0})\rangle_- \\ &= {}_+\langle \alpha e_{x_0} + a_2, \beta e_{x_0} + b_1\rangle_- \\ &= \alpha\beta + {}_+\langle a_2, b_1\rangle_-. \end{aligned} \tag{3.41}$$

The proof is completed if we could show that ${}_{+}\langle a_2, b_1\rangle_{-} = 0$. Notice that a_2 is supported on R_2 and b_1 on R_1, and $R_1 \cap R_2 = \emptyset$. Thus it seems that ${}_{+}\langle a_2, b_1\rangle_{-} = 0$, but we need to confirm it.

Step 2: The case when A is strictly positive. Suppose that there exist $c_1,\ c_2 > 0$ such that $c_1 I \le A \le c_2 I$. In this case A has a bounded inverse A^{-1} in $\mathcal{H}_0 = l^2(E)$. The RK $B(x, y)$ for $\mathcal{H}_-$ (see (2.16)) is given by

$$B(x, y) = {}_{+}\langle e_x, A^{-1}e_y\rangle_{-} = (e_x, A^{-1}e_y)_0, \quad x, y \in E. \tag{3.42}$$

Moreover, as for the elements, the inclusions in (2.5) now become the equalities and the dual pairings in (2.6) and (2.8) are just the inner product in the center space $\mathcal{H}_0$:

$${}_{-}\langle f, g\rangle_{+} = (f, g)_0 = \overline{(g, f)_0} = \overline{{}_{+}\langle g, f\rangle_{-}}, \quad f, g \in \mathcal{H}_0 = \mathcal{H}_- = \mathcal{H}_+, \tag{3.43}$$

the equalities $\mathcal{H}_0 = \mathcal{H}_- = \mathcal{H}_+$ meaning that all the spaces have the same elements. We will, however, keep the pairing notations ${}_{-}\langle\cdot,\cdot\rangle_{+}$ and ${}_{+}\langle\cdot,\cdot\rangle_{-}$ for a convenience. Now let us come back to the Eq. (3.41). The dual pairing is just an inner product in $\mathcal{H}_0$ and the vector a_2 vanishes on R_1 and b_1 lives only on R_1. We therefore have

$${}_{+}\langle a_2, b_1\rangle_{-} = (a_2, b_1)_0 = 0. \tag{3.44}$$

From (3.41) and (3.44) we have $\alpha\beta = 1$.

Step 3: The case when one of R_1 and R_2 is finite. In this case either a_2 in (3.33) or b_1 in (3.40) is finitely supported. Moreover, since they have disjoint supports, by (2.12) we have

$${}_{+}\langle a_2, b_1\rangle_{-} = \sum_{x\in E} \overline{a_2(x)} b_1(x) = 0. \tag{3.45}$$

This, together with (3.41), proves the theorem. This observation gives us more information. Notice that the number β in (2.19) is not altered even if we considered the restriction of B to the set $\widetilde{R_2} := \{x_0\} \cup R_2$. Recall the notation $\|\cdot\|_{\widetilde{R_2};B}$ for the norm in the RKHS $\mathcal{H}_{\widetilde{R_2};B}$ consisting of all the restrictions of vectors in $\mathcal{H}_-$ to the set $\widetilde{R_2}$, and having RK $B_{\widetilde{R_2}}(x, y)$, the restriction of $B(x, y)$ onto $\widetilde{R_2}$. We consider $\widetilde{R_2}$ being partitioned into $\widetilde{R_2} = \{e_{x_0}\} \cup \emptyset \cup R_2$, and then apply the result in this step to get

$$\beta^{-1} = \|e_{x_0}\|^2_{\widetilde{R_2};B} = (e_{x_0}, (B_{\widetilde{R_2}})^{-1}e_{x_0})_0. \tag{3.46}$$

For each $\varepsilon > 0$, we introduce the strictly positive and bounded operators $A(\varepsilon) := A + \varepsilon$ and $B(\varepsilon) := A(\varepsilon)^{-1}$ on $\mathcal{H}_0$. Let $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ be the numbers defined as in (2.19) by replacing the operators A and B with $A(\varepsilon)$ and $B(\varepsilon)$, respectively. By the result in Step 2, we have

$$\alpha(\varepsilon)\beta(\varepsilon) = 1, \quad \varepsilon > 0. \tag{3.47}$$

On the other hand, by (3.46) we have

$$\beta(\varepsilon)^{-1} = \|e_{x_0}\|^2_{\widetilde{R_2};B(\varepsilon)} = (e_{x_0}, (B(\varepsilon)_{\widetilde{R_2}})^{-1} e_{x_0})_0. \tag{3.48}$$

In Lemma 3.7, we have shown that

$$\lim_{\varepsilon\to 0} \|e_{x_0}\|^2_{\widetilde{R_2};B(\varepsilon)} = \|e_{x_0}\|^2_{\widetilde{R_2};B}, \tag{3.49}$$

that is

$$\lim_{\varepsilon\to 0} \beta(\varepsilon)^{-1} = \beta^{-1}. \tag{3.50}$$

It is easy to check that

$$\lim_{\varepsilon\to 0} \alpha(\varepsilon) = \alpha. \tag{3.51}$$

We thus get by (3.47), (3.50)–(3.51), $\alpha\beta = 1$. The proof is completed. □

Remark 3.8. We extend the formula in Proposition 3.1(b) to the infinite system. For each $\Delta \subset E$, as before, we let $A_\Delta(x, y)$ be the restriction of $A(x, y)$ to the set Δ. Then for each $x_0 \in \Delta$, the function $A_\Delta(\cdot, x_0)$ (also denoted by $A(\Delta, x_0)$) belongs to the RKHS $\mathcal{H}_{\Delta;A}$ consisting of all the restrictions of vectors in $\mathcal{H}_+$ to the set Δ and having the RK $A_\Delta(x, y)$ (Theorem 3.2). By Proposition 3.1(b) we see that for each $\Lambda \Subset E$,

$$\begin{aligned}\alpha_\Lambda &= A(x_0, x_0) - A(x_0, \Lambda_1)A(\Lambda_1, \Lambda_1)^{-1}A(\Lambda_1, x_0)\\ &= A(x_0, x_0) - \|A(\Lambda_1, x_0)\|^2_{\Lambda_1;A}.\end{aligned} \tag{3.52}$$

On the other hand, as Λ increases, we have by Theorem 3.3, $\lim_{\Lambda\uparrow E} \|A(\Lambda_1, x_0)\|^2_{\Lambda_1;A} = \|A(R_1, x_0)\|^2_{R_1;A}$. Thus we get

$$\alpha = A(x_0, x_0) - \|A(R_1, x_0)\|^2_{R_1;A}. \tag{3.53}$$

In particular, if A is bounded away from 0, then we we have

$$\alpha = A(x_0, x_0) - A(x_0, R_1)A(R_1, R_1)^{-1}A(R_1, x_0). \tag{3.54}$$

4. PROOFS OF THEOREM 2.6 AND THEOREM 2.7

The proof of Theorem 2.6 will follow from the variational principle of Theorem 2.4 and the projection-inversion inequalities, which we now introduce. For a matrix A on E, we denote by A_Λ the submatrix, or the restriction of A on the set $\Lambda \subset E$.

Lemma 4.1. *Let $A(x, y)$ and $B(x, y)$, $x, y \in E$, be the RK's respectively for $\mathcal{H}_+$ and $\mathcal{H}_-$ in Sec. 2. Then, for any finite subsets $\Lambda \subset \Delta \Subset E$, the following inequalities hold:*

(a) $(A_\Lambda)^{-1} \leq ((A_\Delta)^{-1})_\Lambda \leq B_\Lambda$;
(b) $(B_\Lambda)^{-1} \leq ((B_\Delta)^{-1})_\Lambda \leq A_\Lambda$.

For a proof we need the projection-inversion lemma (see Ref. [7, p. 18], [13, Corollary 5.3], and [3, Lemma A.5]):

Lemma 4.2. *Let T be any bounded positive definite operator with bounded inverse T^{-1}. Then for any projection P,*

$$P(PTP)^{-1}P \leq PT^{-1}P. \tag{4.1}$$

Proof of Lemma 4.1: The first inequalities in (a) and (b) follow from Lemma 4.2. In order to prove the second inequalities it is enough to show $(A_\Lambda)^{-1} \leq B_\Lambda$ and $(B_\Lambda)^{-1} \leq A_\Lambda$, because $(B_\Delta)_\Lambda = B_\Lambda$ and $(A_\Delta)_\Lambda = A_\Lambda$ for $\Lambda \subset \Delta \Subset E$. Moreover, since the matrices are positive definite, either one of the inequalities $(A_\Lambda)^{-1} \leq B_\Lambda$ or $(B_\Lambda)^{-1} \leq A_\Lambda$ implies the other. The inequality $(B_\Lambda)^{-1} \leq A_\Lambda$ has been already shown in (3.16). □

Remark 4.3. Notice that B is formally the inverse of A (see (2.16)), and that the operator A may have 0 in its spectrum (it should then be a continuous spectrum). In that case, the operator B, considered on the space $\mathcal{H}_0$, is an unbounded operator. Therefore, Lemma 4.1 extends Lemma 4.2.

Next we discuss the order relations between the restricted operators and the interaction operators giving the local probability densities of DPP's. For each finite set $\Lambda \subset E$, we let, as before,

$$A_\Lambda := P_\Lambda A P_\Lambda \quad \text{and} \quad A_{[\Lambda]} := K_\Lambda (I - K_\Lambda)^{-1}, \tag{4.2}$$

where P_Λ is the projection on $\mathcal{H}_0 = l^2(E)$ onto $l^2(\Lambda)$ and $K := A(I + A)^{-1}$. We define

$$B_{[\Lambda]} := (A_{[\Lambda]})^{-1} \tag{4.3}$$

and recall that B is the inverse of A.

Lemma 4.3. *For any finite set $\Lambda \subset E$,*

$$A_{[\Lambda]} \leq A_\Lambda \quad \text{and} \quad B_{[\Lambda]} \leq B_\Lambda. \tag{4.4}$$

Proof. We first prove the inequality $A_{[\Lambda]} \le A_\Lambda$. By Lemma 4.2,

$$\begin{aligned} A_{[\Lambda]} &= -I_\Lambda + ((I-K)_\Lambda)^{-1} \\ &\le -I_\Lambda + P_\Lambda (I-K)^{-1} P_\Lambda \\ &= A_\Lambda, \end{aligned} \tag{4.5}$$

where $I_\Lambda := P_\Lambda I P_\Lambda$. The second inequality in (4.4) can be shown in two ways. We introduce both of them. First, as before, we define $A(\varepsilon) = A + \varepsilon$ and $B(\varepsilon) = A(\varepsilon)^{-1}$ for $\varepsilon > 0$. By the same way used in (4.5) we can show

$$B(\varepsilon)_{[\Lambda]} := (A(\varepsilon)_{[\Lambda]})^{-1} \le B(\varepsilon)_\Lambda, \tag{4.6}$$

where $A(\varepsilon)_{[\Lambda]} := K(\varepsilon)_\Lambda (I - K(\varepsilon)_\Lambda)^{-1}$ with $K(\varepsilon) := A(\varepsilon)(I + A(\varepsilon))^{-1}$. Since $K(\varepsilon) \to K$ uniformly as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon \to 0} B(\varepsilon)_{[\Lambda]} = (A_{[\Lambda]})^{-1} = B_{[\Lambda]}. \tag{4.7}$$

On the other hand, by (3.26)

$$B(\varepsilon)_\Lambda \to B_\Lambda \quad \text{uniformly as } \varepsilon \to 0. \tag{4.8}$$

The inequality $B_{[\Lambda]} \le B_\Lambda$ follows from (4.6)–(4.8).

The second way is to use Lemma 4.1. $B_{[\Lambda]}$ can be rewritten as $B_{[\Lambda]} = -I_\Lambda + (K_\Lambda)^{-1}$. Since $K = A(I + A)^{-1}$, K satisfies the conditions in the hypothesis (H) of Sec. 2. Applying Lemma 4.1(a) for the pair of operators K and K^{-1}, we have

$$\begin{aligned} B_{[\Lambda]} &\le -I_\Lambda + P_\Lambda K^{-1} P_\Lambda \\ &= P_\Lambda(-I + K^{-1})P_\Lambda \\ &= P_\Lambda A^{-1} P_\Lambda = B_\Lambda. \end{aligned}$$

The proof is completed.

We are now ready to prove Theorem 2.6

Proof of Theorem 2.6: Let $x_0 \in E$ and $\xi \in \mathcal{X}$ be any configuration with $x_0 \notin \xi$. For a convenience we define an auxiliary configuration $\overline{\xi} \in \mathcal{X}$ as

$$\overline{\xi} := E \setminus (\xi \cup \{x_0\}). \tag{4.9}$$

From the definition (2.21) and Proposition 3.1(b) we have the equality:

$$\alpha_{[\Lambda]} = (A_{[\Lambda]}(x_0\xi_\Lambda, x_0\xi_\Lambda)^{-1}(x_0, x_0))^{-1}. \tag{4.10}$$

By the first inequality in Lemma 4.4 and using Proposition 3.1 once more we have the bound

$$\alpha_{[\Lambda]} \le (A_\Lambda(x_0\xi_\Lambda, x_0\xi_\Lambda)^{-1}(x_0, x_0))^{-1} = \alpha_\Lambda, \tag{4.11}$$

where α_Λ is defined in (2.18) with $R_1 := \xi$ (and $\Lambda_1 = \Lambda \cap R_1 = \Lambda \cap \xi \equiv \xi_\Lambda$). Now by Proposition 3.1(b) and (c) we have

$$\beta_{[\Lambda]} := (\alpha_{[\Lambda]})^{-1} = (B_{[\Lambda]}(x_0\overline{\xi}_\Lambda, x_0\overline{\xi}_\Lambda)^{-1}(x_0, x_0))^{-1}, \tag{4.12}$$

where $B_{[\Lambda]} = (A_{[\Lambda]})^{-1}$. By the second inequality of Lemma 4.4 we also have the bound

$$\beta_{[\Lambda]} \leq \beta_\Lambda, \tag{4.13}$$

where, again, β_Λ is defined in (2.18) with $R_2 := \overline{\xi}$. Now we take the limit of Λ increasing to the whole space E. Since $\alpha_\Lambda \to \alpha$ as Λ increases to E we have from (4.11)

$$\limsup_{\Lambda \uparrow E} \alpha_{[\Lambda]} \leq \alpha. \tag{4.14}$$

On the other hand, since $\beta_\Lambda \to \beta$ as $\Lambda \uparrow E$, we have also from (4.13)

$$\liminf_{\Lambda \uparrow E} \alpha_{[\Lambda]} = (\limsup_{\Lambda \uparrow E} \beta_{[\Lambda]})^{-1} \geq \beta^{-1} = \alpha. \tag{4.15}$$

The last equality comes from Theorem 2.4. From (4.14) and (4.15) we get $\lim_{\Lambda \uparrow E} \alpha_{[\Lambda]} = \alpha$, which was to be shown. □

Let us now turn to the proof of Theorem 2.7. For the proof of Gibbsianness we will follow the method developed in Ref. 16 for continuum models. We will first define a Gibbsian specification[2,8] by introducing an interaction. Then we will prove that the DPP of our concern is admitted to the specification. We refer also to Ref. [13, Sec. 6]. The proof of uniqueness will be shown following the method of Ref. 13.

Let A be an operator that satisfies the conditions in the hypothesis (H). For any finite configuration $\xi \in \mathcal{X}$, we define an interaction potential of the particles in ξ by[16]

$$V(\xi) := -\log \det A(\xi, \xi). \tag{4.16}$$

Notice that $V(\xi) < \infty$ for all finite configurations $\xi \in \mathcal{X}$. For any $\Lambda_1, \Lambda_2 \Subset E$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$, and for any configurations ξ_{Λ_1} and ξ_{Λ_2} on the sets Λ_1 and Λ_2, respectively, the mutual potential energy $W(\xi_{\Lambda_1}; \xi_{\Lambda_2})$ is defined to satisfy

$$V(\xi_{\Lambda_1} \cup \xi_{\Lambda_2}) = V(\xi_{\Lambda_1}) + V(\xi_{\Lambda_2}) + W(\xi_{\Lambda_1}; \xi_{\Lambda_2}). \tag{4.17}$$

Now for each $\zeta_\Lambda \in \mathcal{X}_\Lambda$ and $\xi \in \mathcal{X}$, we define the energy of the particle configuration ζ_Λ on Λ with boundary condition ξ by

$$H_\Lambda(\zeta_\Lambda; \xi) := \lim_{\Delta \uparrow E}(V(\zeta_\Lambda) + W(\zeta_\Lambda; \xi_{\Delta \setminus \Lambda})), \tag{4.18}$$

whenever the limit exists. As a matter of fact, $H_\Lambda(\zeta_\Lambda;\xi)$ is well-defined for all $\zeta_\Lambda \in \mathcal{X}_\Lambda$ and $\xi \in \mathcal{X}$ as shown in the following lemma:

Lemma 4.5. *Suppose that the operator A satisfies the conditions in the hypothesis (H). Then for any $\zeta_\Lambda \in \mathcal{X}_\Lambda$ and $\xi \in \mathcal{X}$, the value $H_\Lambda(\zeta_\Lambda;\xi)$ in (4.18) is well-defined as a finite number.*

Proof. We define first for each bounded set $\Delta \supset \Lambda$

$$H_{\Lambda;\Delta}(\zeta_\Lambda;\xi) := V(\zeta_\Lambda) + W(\zeta_\Lambda;\xi_{\Delta\setminus\Lambda}). \tag{4.19}$$

From the definitions (4.16)–(4.17) we get

$$H_{\Lambda;\Delta}(\zeta_\Lambda;\xi) = -\log \frac{\det A(\zeta_\Lambda\xi_{\Delta\setminus\Lambda}, \zeta_\Lambda\xi_{\Delta\setminus\Lambda})}{\det A(\xi_{\Delta\setminus\Lambda}, \xi_{\Delta\setminus\Lambda})}. \tag{4.20}$$

Let $\zeta_\Lambda = \{x_1, \ldots, x_n\}$ be an enumeration of the sites in ζ_Λ. Then we can rewrite the quantity inside the logarithm in (4.20) as

$$\begin{aligned} &\frac{\det A(\zeta_\Lambda\xi_{\Delta\setminus\Lambda}, \zeta_\Lambda\xi_{\Delta\setminus\Lambda})}{\det A(\xi_{\Delta\setminus\Lambda}, \xi_{\Delta\setminus\Lambda})} \\ &= \frac{\det A(x_1, \ldots, x_n\xi_{\Delta\setminus\Lambda}, x_1, \ldots, x_n\xi_{\Delta\setminus\Lambda})}{\det A(x_2, \ldots, x_n\xi_{\Delta\setminus\Lambda}, x_2, \ldots, x_n\xi_{\Delta\setminus\Lambda})} \cdots \frac{\det A(x_n\xi_{\Delta\setminus\Lambda}, x_n\xi_{\Delta\setminus\Lambda})}{\det A(\xi_{\Delta\setminus\Lambda}, \xi_{\Delta\setminus\Lambda})}. \end{aligned} \tag{4.21}$$

By Theorem 2.4, each term in the r.h.s. converges to a strictly positive number as Δ increases to E. The proof is complete. □

Let us now define the Gibbsian specification. Define a partition function on the set Λ with a boundary condition $\xi \in \mathcal{X}$ as

$$Z_\Lambda(\xi) := \sum_{\zeta_\Lambda \subset \Lambda} \exp[-H_\Lambda(\zeta_\Lambda;\xi)]. \tag{4.22}$$

Then we define a probability distribution on the particle configurations as

$$\gamma_\Lambda(\zeta_\Lambda;\xi) := \frac{1}{Z_\Lambda(\xi)} \exp[-H_\Lambda(\zeta_\Lambda;\xi)]. \tag{4.23}$$

Let the set $\{0, 1\}$ be equipped with a discrete topology and $\Omega := \{0, 1\}^E$ with a product topology. Let $\mathcal{F}$ be the Borel σ-algebra on Ω. For any subset $\Delta \subset E$ we let $\mathcal{F}_\Delta$ be the σ-algebra on Ω such that the map ξ_x is measurable for all $x \in \Delta$. We notice that $\mathcal{F}_E = \mathcal{F}$. By the natural mapping between Ω and $\mathcal{X}$, we define σ-algebras $\mathcal{F}_\Delta$, $\Delta \subset E$, and $\mathcal{F}$ on $\mathcal{X}$. The Gibbsian specification is defined as follows:[2,8] for any measurable set $A \in \mathcal{F}$ and $\xi \in \mathcal{X}$, we define

$$\gamma_\Lambda(A|\xi) := \sum_{\zeta_\Lambda \subset \Lambda} \gamma_\Lambda(\zeta_\Lambda; \xi) 1_A(\zeta_\Lambda \xi_{\Lambda^c}), \tag{4.24}$$

where 1_A denotes the indicator function on the set A. It is not hard to check that the system $(\gamma_\Lambda)_{\Lambda \Subset E}$ defines a specification, i.e., it satisfies the following properties:

(i) $\gamma_\Lambda(\cdot|\xi)$ is a probability measure for each $\xi \in \mathcal{X}$;
(ii) $\gamma_\Lambda(A|\cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable for all $A \in \mathcal{F}$;
(iii) $\gamma_\Lambda(A|\cdot) = 1_A(\cdot)$ if $A \in \mathcal{F}_{\Lambda^c}$;
(iv) $\gamma_\Delta \gamma_\Lambda(A|\xi) := \sum_{\zeta_\Delta \subset \Delta} \gamma_\Delta(\zeta_\Delta|\xi)\gamma_\Lambda(A|\zeta_\Delta \xi_{\Delta^c}) = \gamma_\Delta(A|\xi)$ for all $\Lambda \subset \Delta \Subset E$ and $\xi \in \mathcal{X}$.

A probability measure μ on $(\mathcal{X}, \mathcal{F})$ is said to be *admitted* to the specification $(\gamma_\Lambda)_{\Lambda \Subset E}$, or a *Gibbs measure*, if it satisfies the DLR-equations:

$$\mu(A) = \int \gamma_\Lambda(A|\xi) d\mu(\xi), \quad \text{for any} \quad A \in \mathcal{F} \quad \text{and} \quad \Lambda \Subset E. \tag{4.25}$$

The DLR condition says that for any $\Lambda \Subset E$ and $A \in \mathcal{F}$, the conditional expectation $E^\mu[1_A|\mathcal{F}_{\Lambda^c}]$ has a version $\gamma_\Lambda(A|\cdot)$:

$$E^\mu[1_A|\mathcal{F}_{\Lambda^c}](\xi) = \gamma_\Lambda(A|\xi), \quad \mu\text{–a.a. } \xi. \tag{4.26}$$

From the Eq. (4.21) we easily see that for any $\Lambda \Subset E$, $\zeta_\Lambda \equiv \{x_1, \ldots, x_n\} \in \mathcal{X}_\Lambda$, and $\xi \in \mathcal{X}$,

$$\begin{aligned} &H_\Lambda(\zeta_\Lambda; \xi) \\ &= H_{\{x_1\}}(\{x_1\}; \xi) + H_{\{x_2\}}(\{x_2\}; \{x_1\} \cup \xi) + H_{\{x_n\}}(\{x_n\}; \{x_1, \ldots, x_{n-1}\} \cup \xi). \end{aligned}$$

This says that all the values $H_\Lambda(\zeta_\Lambda; \xi)$ are determined by the values $H_{\{x\}}(\{x\}; \xi)$. Now then the DLR condition (4.26) is equivalent to saying that (cf. Refs. 11 and [13, Sec. 6])

$$\frac{E^\mu[\xi_x = \{x\}|\mathcal{F}_{\{x\}^c}](\xi)}{E^\mu[\xi_x = \emptyset|\mathcal{F}_{\{x\}^c}](\xi)} = \exp[-H_{\{x\}}(\{x\}; \xi)], \quad \forall\, x \in E. \tag{4.27}$$

Proof of Theorem 2.7: *Gibbsianness*. As noted above, it is enough to show the relation (4.27). Let $x_0 \in E$ be a fixed point and let $\xi \in \mathcal{X}$. Then by (4.18)–(4.20),

$$\exp[-H_{\{x_0\}}(\{x_0\}; \xi)] = \lim_{\Delta \uparrow E} \frac{\det A(x_0 \xi_{\Delta \setminus \{x_0\}}, x_0 \xi_{\Delta \setminus \{x_0\}})}{\det A(\xi_{\Delta \setminus \{x_0\}}, \xi_{\Delta \setminus \{x_0\}})}. \tag{4.28}$$

On the other hand, by (2.20)–(2.21)

$$\frac{E^\mu[\xi_{\{x_0\}}=\{x_0\}|\mathcal{F}_{\{x_0\}^c}](\xi)}{E^\mu[\xi_{\{x_0\}}=\emptyset|\mathcal{F}_{\{x_0\}^c}](\xi)} = \lim_{\Delta\uparrow E}\frac{E^\mu[\xi_{\{x_0\}}=\{x_0\}|\mathcal{F}_{\Delta\setminus\{x_0\}}](\xi_{\Delta\setminus\{x_0\}})}{E^\mu[\xi_{\{x_0\}}=\emptyset|\mathcal{F}_{\Delta\setminus\{x_0\}}](\xi_{\Delta\setminus\{x_0\}})}$$
$$= \lim_{\Delta\uparrow E}\frac{\det A_{[\Delta]}(x_0\xi_{\Delta\setminus\{x_0\}}, x_0\xi_{\Delta\setminus\{x_0\}})}{\det A_{[\Delta]}(\xi_{\Delta\setminus\{x_0\}}, \xi_{\Delta\setminus\{x_0\}})}. \tag{4.29}$$

By Theorem 2.6 the two limits in (4.28) and (4.29) are the same and this proves that the DPP μ corresponding to the operator $A(I+A)^{-1}$ is a Gibbs measure admitted to the specification $(\gamma_\Lambda)_{\Lambda\Subset E}$ in (4.23)–(4.24).

Uniqueness. Let us now address to the uniqueness of the Gibbs measure. The arguments in the sequel parallel those in Ref. [13, Sec. 6]. Suppose that ν is a probability measure admitted to the specification $(\gamma_\Lambda)_{\Lambda\Subset E}$, i.e., ν satisfies the condition (4.26):

$$E^\nu[1_A|\mathcal{F}_{\Lambda^c}](\xi) = \gamma_\Lambda(A|\xi), \quad \nu\text{−a.a.}\ \xi\in\mathcal{X} \quad \text{for all} \quad \Lambda\Subset E. \tag{4.30}$$

Let $F:\mathcal{X}\to\mathbb{R}$ be a function of the form

$$F(\xi) = 1_{\{\xi_{\Lambda_0}=X\}}, \quad \text{for some} \quad \Lambda_0\Subset E \quad \text{and} \quad X\subset\Lambda_0. \tag{4.31}$$

We will show that for such functions F,

$$\nu(F) = \mu(F). \tag{4.32}$$

Since those functions F generate the σ-algebra $\mathcal{F}$, ν then should be μ and the uniqueness follows.

Let $\Lambda\Subset E$ be any set with $\Lambda_0\subset\Lambda$. Then by (4.30)

$$E^\nu[F|\mathcal{F}_{\Lambda^c}](\xi) = \frac{1}{Z_\Lambda(\xi)}\sum_{Y\subset\Lambda\setminus\Lambda_0}\exp[-H_\Lambda(X\cup Y;\xi)]. \tag{4.33}$$

Notice that the partition function $Z_\Lambda(\xi)$ can be rewritten as follows. Let $\Phi^{(\Lambda;\xi)}$ be a matrix of size $|\Lambda|$ whose components are given by

$$\Phi^{(\Lambda;\xi)}(x,y) := A(x,y) - (A_{\xi_{\Lambda^c}}(\cdot,x), A_{\xi_{\Lambda^c}}(\cdot,y))_{\xi_{\Lambda^c};A}, \tag{4.34}$$

where, as before, $A_{\xi_{\Lambda^c}}(x,y)$ is the restriction of $A(x,y)$ to the set ξ_{Λ^c} and $(\cdot,\cdot)_{\xi_{\Lambda^c};A}$ is the inner product of the RKHS $\mathcal{H}_{\xi_{\Lambda^c};A}$ having RK $A_{\xi_{\Lambda^c}}(x,y)$. By Theorem 3.2, the matrix $\Phi^{(\Lambda;\xi)}$ is well-defined. In an informal level, we can write $\Phi^{(\Lambda;\xi)}(x,y)$ as

$$\Phi^{(\Lambda;\xi)}(x,y) = A(x,y) - A(x,\xi_{\Lambda^c})A(\xi_{\Lambda^c},\xi_{\Lambda^c})^{-1}A(\xi_{\Lambda^c},y). \tag{4.35}$$

We refer to Ref. [13, p. 1559] for the same matrix, where A is strictly positive. For each finite $\Delta\supset\Lambda$, we let

$$\Phi^{(\Lambda,\Delta;\xi)}(x,y) := A(x,y) - (A_{\xi_{\Delta\setminus\Lambda}}(\cdot,x), A_{\xi_{\Delta\setminus\Lambda}}(\cdot,y))_{\xi_{\Delta\setminus\Lambda};A}, \quad x,y\in\Lambda.$$

By Theorem 3.3,

$$\lim_{\Delta\uparrow E}\Phi^{(\Lambda,\Delta;\xi)}(x,y)=\Phi^{(\Lambda;\xi)}(x,y),\quad x,y\in\Lambda. \tag{4.36}$$

Moreover, it is obvious that for any $X\subset\Lambda$

$$\exp[-H_{\Lambda,\Delta}(X;\xi)]=\det(\Phi^{(\Lambda,\Delta;\xi)}(X,X)) \tag{4.37}$$

and hence

$$\exp[-H_{\Lambda}(X;\xi)]=\det(\Phi^{(\Lambda;\xi)}(X,X)). \tag{4.38}$$

Therefore we get

$$Z_\Lambda(\xi)=\sum_{X\subset\Lambda}\exp[-H_\Lambda(X;\xi)]=\sum_{X\subset\Lambda}\det(\Phi^{(\Lambda;\xi)}(X,X))=\det(I+\Phi^{(\Lambda;\xi)}). \tag{4.39}$$

By using the expression (4.38) we see that (P_Δ denotes the projection on $\mathcal{H}_0$ onto $l^2(\Delta)$)

$$\begin{aligned}\sum_{Y\subset\Lambda\setminus\Lambda_0}\exp[-H_\Lambda(X\cup Y;\xi)]&=\sum_{Y\subset\Lambda\setminus\Lambda_0}\det(\Phi^{(\Lambda;\xi)}_{X\cup Y})\\&=\det(P_{\Lambda\setminus\Lambda_0}+\Phi^{(\Lambda;\xi)}_{X\cup(\Lambda\setminus\Lambda_0)}).\end{aligned} \tag{4.40}$$

Here we have put $\Phi^{(\Lambda;\xi)}_{X\cup Y}\equiv\Phi^{(\Lambda;\xi)}(X\cup Y,X\cup Y)$, etc. We insert (4.39)–(4.40) into the r.h.s. of (4.33) and after a short computation we obtain the expression for $E^\nu[F|\mathcal{F}_{\Lambda^c}](\xi)$ in (4.33) (see Ref. [13, Eq. (6.47)] for the details):

$$\begin{aligned}&E^\nu[F|\mathcal{F}_{\Lambda^c}](\xi)\\&=\frac{\det(P_X[\Phi^{(\Lambda;\xi)}_{\Lambda_0}-\Phi^{(\Lambda;\xi)}(\Lambda_0,\Lambda\setminus\Lambda_0)(I+\Phi^{(\Lambda;\xi)}_{\Lambda\setminus\Lambda_0})^{-1}\Phi^{(\Lambda;\xi)}(\Lambda\setminus\Lambda_0,X)]P_X)}{\det(I+\Phi^{(\Lambda;\xi)}_{\Lambda_0}-\Phi^{(\Lambda;\xi)}(\Lambda_0,\Lambda\setminus\Lambda_0)(I+\Phi^{(\Lambda;\xi)}_{\Lambda\setminus\Lambda_0})^{-1}\Phi^{(\Lambda;\xi)}(\Lambda\setminus\Lambda_0,\Lambda_0))}.\end{aligned} \tag{4.41}$$

We will show that

$$\begin{aligned}&\lim_{\Lambda\uparrow E}[(I+\Phi^{(\Lambda;\xi)})_{\Lambda_0}-\Phi^{(\Lambda;\xi)}(\Lambda_0,\Lambda\setminus\Lambda_0)(I+\Phi^{(\Lambda;\xi)}_{\Lambda\setminus\Lambda_0})^{-1}\Phi^{(\Lambda;\xi)}(\Lambda\setminus\Lambda_0,\Lambda_0)]\\&=(I+A)_{\Lambda_0}-A(\Lambda_0,\Lambda_0^c)(I+A)(\Lambda_0^c,\Lambda_0^c)^{-1}A(\Lambda_0^c,\Lambda_0)\\&=(P_{\Lambda_0}(I+A)^{-1}P_{\Lambda_0})^{-1}.\end{aligned} \tag{4.42}$$

In fact, by using a similar computation as in Proposition 3.1(b) we have for any $f_0\in l^2(\Lambda_0)$,

$$\begin{aligned}&(f_0,[(I+\Phi^{(\Lambda;\xi)})_{\Lambda_0}-\Phi^{(\Lambda;\xi)}(\Lambda_0,\Lambda\setminus\Lambda_0)(I+\Phi^{(\Lambda;\xi)}_{\Lambda\setminus\Lambda_0})^{-1}\\&\times\Phi^{(\Lambda;\xi)}(\Lambda\setminus\Lambda_0,\Lambda_0)]f_0)_{l^2(\Lambda_0)}\end{aligned}$$

$$= \inf_{f \in l^2(\Lambda \setminus \Lambda_0)} (f_0 - f, (I + \Phi^{(\Lambda;\xi)})(\Lambda, \Lambda)(f_0 - f))_{l^2(\Lambda)}$$

$$= \inf_{f \in l^2(\Lambda \setminus \Lambda_0)} (\|f_0 - f\|^2_{l^2(\Lambda)} + \inf_{g \in l^2(\xi_\Lambda^c)} (f_0 - f - g, A(f_0 - f - g))_{l^2(E)})$$

$$= \inf_{h \in l^2(\Lambda_0^c)} (f_0 - h, (P_\Lambda + A)(f_0 - h))_{l^2(E)}. \tag{4.43}$$

Since $P_\Lambda \to I$ strongly as $\Lambda \uparrow E$, it is obvious that the last expression in (4.43) converges as $\Lambda \uparrow E$ to

$$\inf_{h \in l^2(\Lambda_0^c)} (f_0 - h, (I + A)(f_0 - h))_{l^2(E)}$$
$$= (f_0, [(I + A)_{\Lambda_0} - A(\Lambda_0, \Lambda_0^c)(I + A)(\Lambda_0^c, \Lambda_0^c)^{-1} A(\Lambda_0^c, \Lambda_0)] f_0)_{l^2(\Lambda_0)}. \tag{4.44}$$

Eqs. (4.43)–(4.44) prove (4.42). Recall the operator $K = A(I + A)^{-1}$ which gives the DPP μ. We have

$$(I - K)_{\Lambda_0} = P_{\Lambda_0}(I + A)^{-1} P_{\Lambda_0}$$
$$= [(I + A)_{\Lambda_0} - A(\Lambda_0, \Lambda_0^c)(I + A)(\Lambda_0^c, \Lambda_0^c)^{-1} A(\Lambda_0^c, \Lambda_0)]^{-1} \tag{4.45}$$

and

$$A_{[\Lambda_0]} = \frac{K_{\Lambda_0}}{(I - K)_{\Lambda_0}} = (I - K)^{-1}_{\Lambda_0} - I_{\Lambda_0}$$
$$= A(\Lambda_0, \Lambda_0) - A(\Lambda_0, \Lambda_0^c)(I + A)(\Lambda_0^c, \Lambda_0^c)^{-1} A(\Lambda_0^c, \Lambda_0). \tag{4.46}$$

We thus get, by using (4.41)–(4.42) and (4.45)–(4.46),

$$\begin{aligned} \nu(F) &= \lim_{\Lambda \uparrow E} E^\nu[F | \mathcal{F}_{\Lambda^c}](\xi) \\ &= \det(I - K_{\Lambda_0}) \det(P_X A_{[\Lambda_0]} P_X) \\ &= \det(P_X K_{\Lambda_0} + P_{\Lambda_0 \setminus X}(I - K_{\Lambda_0})) \\ &= \mu(F). \end{aligned}$$

Now then ν must be μ and we have proven the uniqueness of the Gibbs measure. $\square$

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APPENDIX

In this Appendix we discuss the hypothesis (H) in Sec. 2 by giving some examples. For simplicity we take $E := \mathbb{Z}$, the set of integers. We give three typical examples.

(i) *The case that A is bounded and has a bounded inverse.* In this case all the norms $\|\cdot\|_-$, $\|\cdot\|_0$, and $\|\cdot\|_+$ are equivalent and the spaces $\mathcal{H}_-$, $\mathcal{H}_0$, and $\mathcal{H}_+$ are the same as sets. Obviously, $\mathcal{H}_-$ is functionally completed. The Gibbsianness of the DPP for the operator $A(I+A)^{-1}$ with A being in this category has already been shown by Shirai and Takahashi.[13]

(ii) *The case of diagonal matrices.* Suppose that A is a diagonal matrix with diagonal elements $\alpha_x > 0$ with α_x being bounded and decreasing to zero as $x \to \infty$. It is not hard to show that the hypothesis (H) is satisfied for those operators A. In fact, $\mathcal{H}_-$ consists of those functions $f : E \to \mathbb{C}$ such that $\sum_{x\in E} \alpha_x |f(x)|^2 < \infty$. In other words, if $g = (g(x))_{x\in E} \in \mathcal{H}_0$ is any element of $\mathcal{H}_0$ then the vector $f \equiv (\alpha_x^{-1/2} g(x))_{x\in E}$ belongs to $\mathcal{H}_-$ and all the elements of $\mathcal{H}_-$ are of this type.

(iii) *Perturbation of diagonal matrices.* Let D be any diagonal matrix of the type in the case (ii) above. Let $A := C^*DC$, where C is a matrix such that C and its inverse C^{-1} have off-diagonal elements that decrease sufficiently fast as the distance from the diagonal become far. To say more concretely, let $C(x, y)$ and $C^{-1}(x, y)$ be the matrix components of C and C^{-1}, respectively. We assume that there exist positive numbers $m > 0$ and $M > 0$ such that

$$m \le C(x, x) \le M \quad \text{and} \quad m \le C^{-1}(x, x) \le M \quad \text{for all} \quad x \in E, \tag{A.1}$$

and $C(x, y)$ and $C^{-1}(x, y)$ converge to zero sufficiently fast as $|x - y| \to \infty$. Then A satisfies the conditions in (H). Here we give an example. Let D be a diagonal matrix with diagonal elements α_x, $x \in E$. We assume that there is $k \in \mathbb{N}$ such that

$$\alpha_x^{-1} \le (1 + |x|)^k, \quad x \in E. \tag{A.2}$$

Let C be a bounded operator with bounded inverse C^{-1} such that there is $m \ge 2(k + 1)$ and

$$|C(x, y)| \le \frac{1}{1 + |x - y|^m} \quad \text{and} \quad |C^{-1}(x, y)| \le \frac{1}{1 + |x - y|^m}. \tag{A.3}$$

Such an operator C can, for example, be obtained by taking its convolution kernel function as the Fourier series of strictly positive and sufficiently smooth function on the circle. We prove that $\mathcal{H}_-$ is functionally completed. It is enough to show that the pre-Hilbert space $(\mathcal{H}_0, \|\cdot\|_-)$ satisfies the conditions (i) and (ii) of Theorem 3.2. First we show that for any $y \in E$, $f(y)$ is continuous in $(\mathcal{H}_0, \|\cdot\|)_-$. As noted in Proposition 2.2(d), it is equivalent to show that $e_y \in \mathcal{H}_+$. But, we have

$$\begin{aligned} \|e_y\|_+^2 = (e_y, A^{-1}e_y)_0 &= ((C^{-1})^* e_y, D^{-1}(C^{-1})^* e_y)_0 \\ &= \sum_{x\in E} \alpha_x^{-1} |(C^{-1})^* e_y(x)|^2 \\ &\le \sum_{x\in E} (1+|x|)^k \frac{1}{(1+|x-y|^m)^2} < \infty. \end{aligned} \tag{A.4}$$

Next, notice that for any $f \in \mathcal{H}_0$,

$$\|f\|_-^2 = (f, C^*DCf)_0 = (Cf, DCf)_0 = (\|Cf\|_-^{(D)})^2, \tag{A.5}$$

where $\|\cdot\|_-^{(D)}$ is the "$-$"-norm for $A \equiv D$. Since we have observed in case (ii) that the space $\mathcal{H}_-^{(D)}$, completion of $\mathcal{H}_0$ w.r.t. $\|\cdot\|_-^{(D)}$-norm, is functionally completed, it is enough to show that given any sequence $\{f_n\} \subset \mathcal{H}_0$ which is $\|\cdot\|_-$-Cauchy and such that $f_n(y) \to 0$ as $n \to \infty$ for all $y \in E$, $Cf_n(y) \to 0$ as $n \to \infty$ for all $y \in E$, because $\{Cf_n\}$ is $\|\cdot\|_-^{(D)}$-Cauchy. We observe that there is a constant $b > 0$ such that

$$|f_n(y)| \le b(1+|y|)^k, \quad y \in E. \tag{A.6}$$

In fact,

$$|f_n(y)| = |(e_y, f_n)_0| \le \|e_y\|_+ \|f_n\|_-. \tag{A.7}$$

Since $\{f_n\}$ is $\|\cdot\|_-$-Cauchy, $\|f_n\|_-$ is bounded uniformly for $n \in \mathbb{N}$. On the other hand from (A.4), it is not hard to see that there exists $b_1 > 0$ such that

$$\|e_y\|_+ \le b_1(1+|y|)^k, \quad y \in E. \tag{A.8}$$

This proves (A.6). Now we have

$$\begin{aligned} Cf_n(y) &= (e_y, Cf_n)_0 \\ &= (C^* e_y, f_n)_0 \\ &= \sum_{x\in E} \overline{C^* e_y(x)} f_n(x) \\ &= \sum_{x\in\partial(y)} \overline{C^* e_y(x)} f_n(x) + \sum_{x\in\partial(y)^c} \overline{C^* e_y(x)} f_n(x), \end{aligned}$$

where $\partial(y)$ is any sufficiently large but finite set containing y. Since $f_n(x) \to 0$ for all $x \in E$, the first term in the last expression converges to 0 as $n \to \infty$. By using (A.3) and (A.6) we have

$$|\sum_{x\in\partial(y)^c} \overline{C^* e_y(x)} f_n(x)| \le b \sum_{x\in\partial(y)^c} \frac{1}{1+|x-y|^m}(1+|x|)^k. \tag{A.9}$$

Once $\partial(y)$ is taken sufficiently large, the quantity in the r.h.s. of (A.9) becomes as much small as we wish. This completes the proof.

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